Ex $T : \mathcal{F}(\mathbb{R};\mathbb{R}) \to M_{2\times 2}$ given by

$$T(f) = \begin{bmatrix} f(0) & f(1) \\ f(2) & f(3) \end{bmatrix}$$

is a linear transformation.

To see that T preserves addition, suppose $f, g \in \mathcal{F}(\mathbb{R};\mathbb{R})$. Then:

$$T(f+g) \overset{\text{defn of } T}{=} \begin{bmatrix} (f+g)(0) & (f+g)(1) \\ (f+g)(2) & (f+g)(3) \end{bmatrix} \overset{\text{defn of } f+g \text{ in } \mathcal{F}(\mathbb{R};\mathbb{R})}{=} \begin{bmatrix} f(0)+g(0) & f(1)+g(1) \\ f(2)+g(2) & f(3)+g(3) \end{bmatrix}$$

$$\overset{\text{defn of addn in } M_{2\times 2}}{=} \begin{bmatrix} f(0) & f(1) \\ f(2) & f(3) \end{bmatrix} + \begin{bmatrix} g(0) & g(1) \\ g(2) & g(3) \end{bmatrix}$$

$$\overset{\text{defn of } T}{=} T(f) + T(g).$$

So: $T(f+g) = T(f) + T(g)$. ✓

To see that T preserves scalar mult, suppose $f \in \mathcal{F}(\mathbb{R};\mathbb{R})$ and $c \in \mathbb{R}$. Then:

$$T(cf) \overset{\text{defn of } T}{=} \begin{bmatrix} (cf)(0) & (cf)(1) \\ (cf)(2) & (cf)(3) \end{bmatrix} \overset{\text{defn of } cf \text{ in } \mathcal{F}(\mathbb{R};\mathbb{R})}{=} \begin{bmatrix} c(f(0)) & c(f(1)) \\ c(f(2)) & c(f(3)) \end{bmatrix}$$

$$\overset{\text{defn of scalar mult in } M_{2\times 2}}{=} c\begin{bmatrix} f(0) & f(1) \\ f(2) & f(3) \end{bmatrix} \overset{\text{defn of } T}{=} cT(f)$$

So: $T(cf) = cT(f)$. ✓