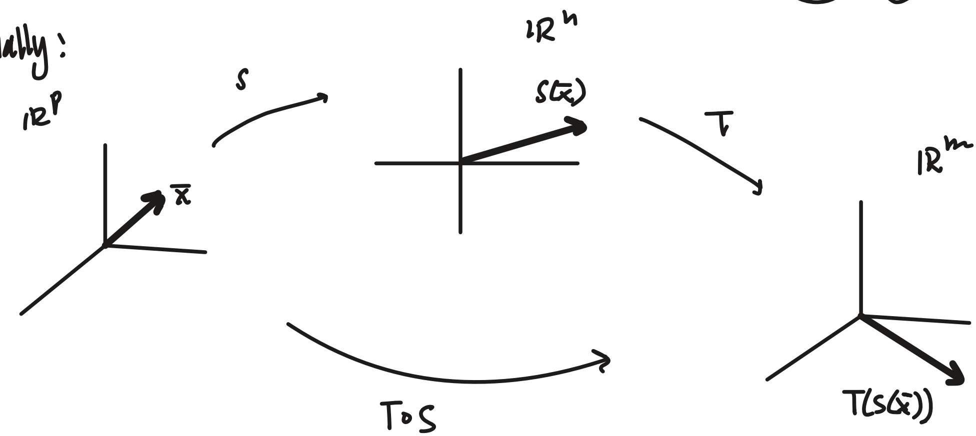
1.9.44. Suppose $S : \mathbb{R}^p \to \mathbb{R}^n$ is a LT.

Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is a LT.

Show that the composition of T with S is a LT.

denoted $\overset{\text{2nd}}{T} \circ \overset{\text{1st}}{S}$

Visually:



(Note: $(T \circ S)(\bar{x})$ means $T(S(\bar{x}))$)

(Here: S and T aren't specific, so we can't build the standard matrices... we have to work with the abstract defn of a LT.)

see p.70

domain of S

Since $S: \mathbb{R}^p \to \mathbb{R}^n$ is a LT, for all $\bar{u}, \bar{v} \in \mathbb{R}^p$ and scalars c,

$$S(\bar{u}+\bar{v}) = S(\bar{u}) + S(\bar{v})$$

and $$S(c\bar{u}) = cS(\bar{u}).$$

everything here lives in $\mathbb{R}^n$

domain of T

Since $T: \mathbb{R}^n \to \mathbb{R}^m$ is a LT, for all $\bar{x}, \bar{y} \in \mathbb{R}^n$ and scalars c,

$$T(\bar{x}+\bar{y}) = \underline{\hspace{3cm}}$$

and $$T(c\bar{x}) = \underline{\hspace{3cm}}$$

fill in the blanks.

Where do these vectors live?

Now, consider $T \circ S : \mathbb{R}^p \to \mathbb{R}^m$. We must show that

$T \circ S$ is linear, i.e. for all $\bar{u}, \bar{v}$ in $\underline{\hspace{1.5cm}}$

and scalars c,

what is the domain of $T \circ S$?

$$(T\circ S)(\bar{u}+\bar{v}) = (T\circ S)(\bar{u}) + (T\circ S)(\bar{v}) \quad ①$$

and $$(T\circ S)(c\bar{u}) = c(T\circ S)(\bar{u}) \quad ②$$

i.e. $$T(S(\bar{u}+\bar{v})) = T(S(u)) + T(S(\bar{v}))$$

and $$T(S(c\bar{u})) = cT(S(\bar{u}))$$

For ①, suppose $\bar{u}, \bar{v} \in \mathbb{R}^p$.

Then $(T \circ S)(\bar{u} + \bar{v}) = T(S(\bar{u} + \bar{v}))$

$= T(\underline{\quad} + \underline{\quad})$ b/c S is linear

$= T(\underline{\quad}) + T(\underline{\quad})$ b/c T is linear

$= (T \circ S)(\bar{u}) + (T \circ S)(\bar{v})$. ✓

For ②, suppose $\bar{u} \in \mathbb{R}^p$ and c is a scalar.

Then $(T \circ S)(c\bar{u}) = T(S(c\bar{u}))$

$= T(\underline{\quad}\ \underline{\quad})$ b/c S is linear

$= \underline{\quad}\, T(\underline{\quad})$ b/c T is linear

$= c(T \circ S)(\bar{u})$. ✓

Since $T \circ S$ satisfies ① and ②, by definition, $T \circ S$ is a LT. ✓