

Math 200 - Linear Algebra  
Exam 1: Practice Exam

Name: Solutions

Please be sure to neatly **show and explain all of your work** and clearly label your answers. This exam is a closed-book, closed-notebook exam. Calculators are not allowed.

Please write and sign the Honor Pledge here when you are done:

Signed:

Problem	Points
1	/12
2	/12
3	/10
4	/12
5	/12
6	/12
7	/10
Total	/80

1. (a) Suppose that a vector  $\vec{b}$  can be written as a linear combination of the vectors

$$\begin{bmatrix} 1 \\ 2 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \\ 5 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \\ 2 \\ 9 \end{bmatrix}.$$

Is the linear combination unique? In other words, is there only one way that the  $\vec{b}$  can be written as a linear combination of these vectors?

We create the matrix  $A$  having these vectors as columns and check whether the matrix has free variables.

$$\begin{bmatrix} 1 & 2 & 4 \\ 2 & 1 & -1 \\ -1 & 0 & 2 \\ 3 & 5 & 9 \end{bmatrix} \begin{array}{l} \textcircled{2} \rightarrow \textcircled{2} - 2\textcircled{1} \\ \textcircled{3} \rightarrow \textcircled{3} + \textcircled{1} \\ \textcircled{4} \rightarrow \textcircled{4} - 3\textcircled{1} \end{array} \begin{bmatrix} 1 & 2 & 4 \\ 0 & -3 & -9 \\ 0 & 2 & 6 \\ 0 & -1 & -3 \end{bmatrix} \begin{array}{l} \textcircled{2} \rightarrow -\frac{1}{3}\textcircled{2} \\ \textcircled{3} \rightarrow \frac{1}{2}\textcircled{3} \\ \textcircled{4} \rightarrow -1\textcircled{4} \end{array} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \\ 0 & 1 & 3 \\ 0 & 1 & 3 \end{bmatrix}$$

Since  $A$  has a free variable, there is more than one way  $\vec{b}$  can be written as a linear combination of these vectors.

$$\begin{array}{l} \textcircled{3} \rightarrow \textcircled{3} - \textcircled{2} \\ \textcircled{4} \rightarrow \textcircled{4} - \textcircled{2} \end{array}$$

$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

- (b) Can every vector in  $\mathbb{R}^4$  be written as a linear combination of the vectors above? Justify your response.

No: For these vectors to span  $\mathbb{R}^4$ ,  $A$  would need to have a pivot in every row. The maximum number of pivots is 3 so it cannot.

↑ free

2. Suppose that

$$A = \begin{bmatrix} 1 & 3 & 1 \\ -4 & -9 & 2 \\ 0 & -3 & -6 \end{bmatrix}$$

Express the solution set of  $A\bar{x} = \bar{0}$  in parametric vector form.

(Note: A closely related question would be to find a collection of vectors that span the solution set of  $A\bar{x} = \bar{0}$ .)

$$\left[ \begin{array}{ccc|c} 1 & 3 & 1 & 0 \\ -4 & -9 & 2 & 0 \\ 0 & -3 & -6 & 0 \end{array} \right] \xrightarrow{\textcircled{2} \rightarrow \textcircled{2} + 4\textcircled{1}} \left[ \begin{array}{ccc|c} 1 & 3 & 1 & 0 \\ 0 & 3 & 6 & 0 \\ 0 & -3 & -6 & 0 \end{array} \right] \xrightarrow{\textcircled{3} \rightarrow \textcircled{2} + \textcircled{3}}$$

$$\left[ \begin{array}{ccc|c} 1 & 0 & -5 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \xrightarrow{\textcircled{1} \rightarrow \textcircled{1} - 2\textcircled{2}} \left[ \begin{array}{ccc|c} 1 & 3 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \xrightarrow{\textcircled{1} \rightarrow \frac{1}{3}\textcircled{2}} \left[ \begin{array}{ccc|c} 1 & 3 & 1 & 0 \\ 0 & 3 & 6 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

These equations say:  $x_1 = 5x_3$

$$x_2 = -2x_3$$

$x_3$  is free

So the soln

(the soln set is spanned by

$$\begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix})$$

set is  $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5x_3 \\ -2x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}$  where  $x_3 \in \mathbb{R}$

3. Please mark each statement as True or False. (Recall that a statement is true if it holds in *all* cases. If the statement fails to hold even once, it is false.) Justify your answer: if the statement is true, give a brief explanation. If the statement is false, give an example that shows that it is false.

(a) For an  $m \times n$  matrix  $A$ , the columns  $\bar{a}_1, \dots, \bar{a}_n$  of  $A$  are linearly dependent if and only if the matrix equation  $A\bar{x} = \bar{0}$  has a nontrivial solution.

True! By defn of matrix multiplication,

$$A\bar{x} = [\bar{a}_1 \dots \bar{a}_n] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1 \bar{a}_1 + \dots + x_n \bar{a}_n$$

So if  $A\bar{x} = \bar{0}$  has a nontrivial solution, this is equivalent to saying  $x_1 \bar{a}_1 + \dots + x_n \bar{a}_n = \bar{0}$  holds for some set  $x_1, \dots, x_n$  not all zero. This is equivalent to saying that the columns  $\bar{a}_1, \dots, \bar{a}_n$  of  $A$  are linearly dependent.

(b) If  $\{\bar{v}_1, \bar{v}_2, \bar{v}_3\}$  is a linearly dependent set, then  $\bar{v}_2 = c\bar{v}_1 + d\bar{v}_3$  for some scalars  $c$  and  $d$ .

False! It is true that for a linearly dependent set  $\{\bar{v}_1, \bar{v}_2, \bar{v}_3\}$  at least one of  $\bar{v}_1, \bar{v}_2, \bar{v}_3$  is a linear combination of the others, but it isn't guaranteed that every  $\bar{v}_1, \bar{v}_2, \bar{v}_3$  is a linear comb of the others. For example:  $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$  is lin dep, but  $\bar{v}_2 \neq c\bar{v}_1 + d\bar{v}_3$  for any  $c, d$ .

4. Suppose that  $A$  is a  $5 \times 5$  matrix and that the linear transformation  $\bar{x} \mapsto A\bar{x}$  is one-to-one. Please give a direct argument, based on attributes of  $A$ , that explains why the map  $\bar{x} \mapsto A\bar{x}$  is onto. As part of your explanation, make sure to say what it means for a linear transformation to be one-to-one and what it means for a linear transformation to be onto.

(Note: please give a full explanation. A simple reference to the Invertible Matrix Theorem will not suffice here.)

To say that  $\bar{x} \mapsto A\bar{x}$  is one-to-one means that for any  $\bar{b}$  for which  $A\bar{x} = \bar{b}$  is consistent, there is a unique  $\bar{x}$  s.t.  $A\bar{x} = \bar{b}$ . Since there is a unique soln,  $A$  cannot have any free variables, so every column must be a pivot column. Thus there are 5 pivots. Since  $A$  is square  $5 \times 5$ , this means there are 5 pivot rows as well, i.e. a pivot in every row. This tells us that for any  $\bar{b}$ , the equation  $A\bar{x} = \bar{b}$  is consistent, i.e. for any  $\bar{b}$ , there is an  $\bar{x}$  s.t.  $\bar{b} = A\bar{x}$ . This is the definition of what it means for the transformation  $\bar{x} \mapsto A\bar{x}$  to be onto.

5. Suppose that  $A$  is a  $4 \times 3$  matrix. Suppose that three vectors  $\bar{v}_1, \bar{v}_2,$  and  $\bar{v}_3$  in the solution set of the equation  $A\bar{x} = \bar{0}$  span  $\mathbb{R}^3$ . Write a short paragraph that justifies why  $A\bar{y} = \bar{0}$  for all  $\bar{y} \in \mathbb{R}^3$ .

(Hint: To begin, what does it mean for a vector  $\bar{v}$  to be in the solution set of the equation  $A\bar{x} = \bar{0}$ ? Now, suppose that  $\bar{y} \in \mathbb{R}^3$ . How can you relate  $\bar{y}$  to  $\bar{v}_1, \bar{v}_2,$  and  $\bar{v}_3$ ?)

$A$  is a  $4 \times 3$  matrix so  $\bar{x} \mapsto A\bar{x}$  maps from  $\mathbb{R}^3$  to  $\mathbb{R}^4$ .

If  $\bar{v}_1, \bar{v}_2, \bar{v}_3$  are in the solution set of  $A\bar{x} = \bar{0}$ , this means  $A\bar{v}_1 = \bar{0}, A\bar{v}_2 = \bar{0}, A\bar{v}_3 = \bar{0}$ .

If  $\bar{v}_1, \bar{v}_2, \bar{v}_3$  span  $\mathbb{R}^3$ , this means every vector in  $\mathbb{R}^3$  can be written as a linear combination of  $\bar{v}_1, \bar{v}_2, \bar{v}_3$ . So, if  $\bar{y} \in \mathbb{R}^3$ ,  $\bar{y} = x_1\bar{v}_1 + x_2\bar{v}_2 + x_3\bar{v}_3$  for some weights  $x_1, x_2, x_3$ .

We want to show  $A\bar{y} = \bar{0}$ . Consider

$$\begin{aligned} A\bar{y} &= A(x_1\bar{v}_1 + x_2\bar{v}_2 + x_3\bar{v}_3) && \text{(substitution)} \\ &= x_1A\bar{v}_1 + x_2A\bar{v}_2 + x_3A\bar{v}_3 && (\bar{x} \mapsto A\bar{x} \text{ is linear}) \\ &= x_1\bar{0} + x_2\bar{0} + x_3\bar{0} && (A\bar{v}_1 = A\bar{v}_2 = A\bar{v}_3 = \bar{0}) \\ &= \bar{0}. && \text{So } A\bar{y} = \bar{0} \text{ for all } \bar{y} \in \mathbb{R}^3. \checkmark \end{aligned}$$

6. Consider the map  $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1^2 \\ x_1 + x_2 \end{bmatrix}$$

Is  $T$  a linear transformation? Either prove that it is or give an example that shows that it is not, using the **definition of a linear transformation**.

For  $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  to be linear, it must be the case that

$$\textcircled{1} T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$$

$$\text{and } \textcircled{2} T(c\vec{u}) = cT(\vec{u}) \text{ for all } \vec{u}, \vec{v} \in \mathbb{R}^2 \text{ and } c \in \mathbb{R}.$$

Here  $T$  is not linear. For example, if  $\vec{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$  and  $\vec{v} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$ , then

$$T(\vec{u}) = T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 1^2 \\ 1+2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \text{ and } T(\vec{v}) = T\left(\begin{bmatrix} 2 \\ 4 \end{bmatrix}\right) = \begin{bmatrix} 2^2 \\ 2+4 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$$

$$\text{Also } T(\vec{u} + \vec{v}) = T\left(\begin{bmatrix} 3 \\ 6 \end{bmatrix}\right) = \begin{bmatrix} 3^2 \\ 3+6 \end{bmatrix} = \begin{bmatrix} 9 \\ 9 \end{bmatrix}. \text{ But } T(\vec{u}) + T(\vec{v}) = \begin{bmatrix} 1 \\ 3 \end{bmatrix} + \begin{bmatrix} 4 \\ 6 \end{bmatrix} = \begin{bmatrix} 5 \\ 9 \end{bmatrix}$$

not equal!

Since  $T(\vec{u} + \vec{v}) \neq T(\vec{u}) + T(\vec{v})$ ,  $T$  can't be linear.

is enough to show  $T$  is not linear, but you could also show the second condition fails.

Another counterexample: Let  $\vec{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$  so  $2\vec{u} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$ .

Then, from above,  $T(\vec{u}) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$  and  $T(2\vec{u}) = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$ . On the other

hand  $2T(\vec{u}) = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$ . Since  $2T(\vec{u}) \neq T(2\vec{u})$ ,  $T$  can't be linear.

$$\begin{bmatrix} 1 & 3 & 4 \\ 0 & 1 & 4 \\ 1 & 1 & 0 \end{bmatrix} \begin{matrix} \\ \\ A \end{matrix} \begin{matrix} \xrightarrow{-1} \\ \xrightarrow{-1} \\ \xrightarrow{-1/4} \end{matrix} \begin{bmatrix} -1 & 1 & 2 \\ 1 & -1 & -1 \\ -1/4 & 1/2 & 1/4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ A^{-1} = I_3$$

7. Compute the inverse of the matrix

$$\begin{bmatrix} 1 & 3 & 4 \\ 0 & 1 & 4 \\ 1 & 1 & 0 \end{bmatrix}$$

(As extra practice, you could multiply your answer by the matrix above to check your work.)

$$\left[ \begin{array}{ccc|ccc} 1 & 3 & 4 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array} \right] \begin{matrix} \textcircled{2} \rightarrow \textcircled{2} - \textcircled{1} \\ \\ \end{matrix} \rightarrow \left[ \begin{array}{ccc|ccc} 1 & 3 & 4 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 0 & -2 & -4 & -1 & 0 & 1 \end{array} \right]$$

$$\left. \begin{matrix} \\ \\ \end{matrix} \right\} \textcircled{3} \rightarrow \textcircled{3} + 2\textcircled{2}$$

$$\left[ \begin{array}{ccc|ccc} 1 & 3 & 0 & 2 & -2 & -1 \\ 0 & 1 & 0 & 1 & -1 & -1 \\ 0 & 0 & 4 & -1 & 2 & 1 \end{array} \right] \begin{matrix} \textcircled{2} \rightarrow \textcircled{2} - \textcircled{3} \\ \textcircled{1} \rightarrow \textcircled{1} - \textcircled{3} \\ \\ \end{matrix} \rightarrow \left[ \begin{array}{ccc|ccc} 1 & 3 & 4 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 0 & 0 & 4 & -1 & 2 & 1 \end{array} \right]$$

$$\left. \begin{matrix} \\ \\ \end{matrix} \right\} \textcircled{1} \rightarrow \textcircled{1} - 3\textcircled{2}$$

$$\left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 2 \\ 0 & 1 & 0 & 1 & -1 & -1 \\ 0 & 0 & 4 & -1 & 2 & 1 \end{array} \right] \xrightarrow{\frac{1}{4}\textcircled{3}} \left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 2 \\ 0 & 1 & 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & -1/4 & 1/2 & 1/4 \end{array} \right] \\ A^{-1} \uparrow$$