Math 200 - Linear Algebra
Exam 2: Practice Exam

Name: solutions

Please be sure to neatly **show and explain all of your work** and clearly label your answers. This exam is a closed-book, closed-notebook exam. Calculators are not allowed.

Please write and sign the Honor Pledge here when you are done:

Signed:

Problem	Points
1	/10
2	/12
3	/12
4	/12
5	/12
6	/12
7	/10
Total	/80

1. Suppose that $T : \mathbb{R}^3 \to \mathbb{R}^3$ is the linear transformation given by $\bar{x} \mapsto A\bar{x}$ where

$$A = \begin{bmatrix} 5 & 1 & 3 \\ 0 & 0 & 2 \\ -4 & -2 & 7 \end{bmatrix}.$$

Let S be a subset of $\mathbb{R}^3$ of volume 6. What is the volume of $T(S)$?

$$\text{vol } T(S) = \det A \,(\text{vol } S) = 6 \det A$$

along 1st row

$$\det A = 5\begin{vmatrix} 0 & 2 \\ -2 & 7 \end{vmatrix} - 1\begin{vmatrix} 0 & 2 \\ -4 & 7 \end{vmatrix} + 3\begin{vmatrix} 0 & 0 \\ -4 & -2 \end{vmatrix}$$

$$= 5(0-(-4)) - 1(0-(-8)) + 3(0-0)$$

$$= 20 - 8 = 12.$$

alternate version... along 2nd row

$$\det A = -0\begin{vmatrix} 1 & 3 \\ -2 & 7 \end{vmatrix} + 0\begin{vmatrix} 5 & 3 \\ -4 & 7 \end{vmatrix} - 2\begin{vmatrix} 5 & 1 \\ -4 & -2 \end{vmatrix}$$

$$= 0 + 0 - 2(-10-(-4))$$

$$= -2(-6) = 12.$$

So $\boxed{\text{vol } T(S) = 6(12) = 72}$

2. Consider the basis

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \end{bmatrix} \right\}$$

for $\mathbb{R}^2$.

(a) Compute the change of basis matrix that transforms vectors written in the standard basis of $\mathbb{R}^2$ to $\mathcal{B}$-coordinate vectors.

$$\bar{x} = c_1\bar{b}_1 + c_2\bar{b}_2 \Rightarrow \bar{x} = \underbrace{\begin{bmatrix} \bar{b}_1 & \bar{b}_2 \end{bmatrix}}_{P_{\mathcal{B}}}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \Rightarrow \bar{x} = P_{\mathcal{B}}[\bar{x}]_{\mathcal{B}}$$

$$\Downarrow$$

$$P_{\mathcal{B}}^{-1}\bar{x} = [\bar{x}]_{\mathcal{B}}.$$

Thus, we want $P_{\mathcal{B}}^{-1}$.

$$P_{\mathcal{B}} = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} \Rightarrow P_{\mathcal{B}}^{-1} = \frac{1}{5-6}\begin{bmatrix} 5 & -2 \\ -3 & 1 \end{bmatrix} = \boxed{\begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix}}$$

(b) Suppose that $\bar{v} = \begin{bmatrix} 1 \\ 6 \end{bmatrix}$. What is $[\bar{v}]_{\mathcal{B}}$.

Using the answer to part (a), we have

$$[\bar{v}]_{\mathcal{B}} = \begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix}\bar{v}$$

$$= \begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 6 \end{bmatrix} = \boxed{\begin{bmatrix} 7 \\ -3 \end{bmatrix}} \leftarrow [\bar{v}]_{\mathcal{B}}$$

We must show: • $\bar{0}_{M_{2\times2}} \in H$

• if $A, B \in H$, then $A+B \in H$ • if $A \in H$, $c \in \mathbb{R}$, then $cA \in H$.

3. Suppose that D is a fixed 4×4 matrix. Let H be the set of all 4×4 matrices that commute with D. In other words,

$$H = \{A \in M_{4\times4} \mid AD = DA\}.$$

Prove that H is a subspace of $M_{4\times4}$.

(Note: the matrix D here is unchanging. We use it to define the set H, but the elements you want to consider for this problem are the matrices A such that $AD = DA$.)

$\bar{0}_{M_{2\times2}}$

1. $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in H$ b/c $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} D = D \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ ← both equal $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

Thus $\boxed{\bar{0}_{M_{2\times2}} \in H}$ ✓

2. Suppose $A, B \in H$. So $AD = DA$ and $BD = DB$.

Then $(A+B)D \overset{\text{prop of matrix mult}}{=} AD + BD \overset{A,B\in H}{=} DA + DB \overset{\text{property of matrix mult}}{=} D(A+B)$

Since $(A+B)D = D(A+B)$, $\boxed{A+B \in H.}$ ✓

3. Suppose $A \in H$ and $c \in \mathbb{R}$. So $AD = DA$.

Then $(cA)D \overset{\text{prop of matrix mult}}{=} c(AD) \overset{A\in H}{=} c(DA) \overset{\text{property of matrix mult}}{=} D(cA)$

Since $(cA)D = D(cA)$, $\boxed{cA \in H.}$ ✓

Thus we conclude that H is a subspace of $M_{2\times2}$.

4. Are the following elements of $\mathbb{P}_2$ linearly independent?

$$\{1+t, 3+2t+3t^2, 1+3t^2\}$$

Please show all work and fully explain your reasoning.

We create coordinate vectors using the basis $\mathcal{B} = \{1, t, t^2\}$ for $\mathbb{P}_2$:

$$[1+t]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad [3+2t+3t^2]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix} \quad [1+3t^2]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$$

and check if the coordinate vectors are linearly indep.

$$\begin{bmatrix} 1 & 3 & 1 \\ 1 & 2 & 0 \\ 0 & 3 & 3 \end{bmatrix} \xrightarrow{②\to②-①} \begin{bmatrix} 1 & 3 & 1 \\ 0 & -1 & -1 \\ 0 & 3 & 3 \end{bmatrix} \xrightarrow{②\to -②} \begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & 1 \\ 0 & 3 & 3 \end{bmatrix} \xrightarrow{③\to③-3②} \begin{bmatrix} \boxed{1} & 3 & 1 \\ 0 & \boxed{1} & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Since the reduced matrix has a free variable, we conclude that the coordinate vectors are not linearly independent. Thus, the original given vectors in $\mathbb{P}_2$ are $\boxed{\text{not linearly independent}}$.

5. Please mark the following statements true or false. If a statement is always true, give a brief justification why it is true. If it is false, give an example that shows that it is false.

(a) If V contains a collection $\{\vec{v}_1, \ldots, \vec{v}_k\}$ that are linearly independent, then $\dim V = k$.

$\boxed{\text{False!}}$ If $\{\vec{v}_1, \ldots, \vec{v}_k\}$ doesn't also span V, then it is not a basis. So $\dim V$ could be less than k.

Ex $\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right\}$ are linearly indep in $\mathbb{R}^3$, but $\dim \mathbb{R}^3 = 3$.

(b) The map $T : M_{2\times 2} \to \mathbb{R}$ given by

$$T(A) = \det A$$

is a linear transformation.

key: while it is true for all A, B that $\det AB = \det A \det B$, in general, $\det(A+B) \neq \det A + \det B$

$\boxed{\text{False!}}$

Consider $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ in $M_{2\times2}$.

Then $T(A+B) = \det(A+B) = \det\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$ — since $T(A+B) \neq T(A)+T(B)$, T is not a L.T.

but $T(A)+T(B) = \det A + \det B = 1+1 = 2$

(c) For the linear transformation $T : \mathcal{F}(\mathbb{R}; \mathbb{R}) \to \mathbb{R}^3$ given by

$$T(f) = \begin{bmatrix} f(1) \\ f(0) \\ f(1) \end{bmatrix},$$

the kernel of T is the set of all functions f such that $f(0) = f(1) = 0$.

$\boxed{\text{True!}}$ The zero vector in $\mathbb{R}^3$ is $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. Thus $\ker T$ is the set of f such that $T(f) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, ie f s.t. $f(1) = f(0) = 0$.

*If you wanted to justify this, suppose

$\bar{p}(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$.

Then $T(\bar{p}) = \bar{p}'(t) = 0 + a_1 + 2a_2 t + 3a_3 t^2$. $\quad \bar{p}'(t) = a_1 + 2a_2 t + 3a_3 t^2$

$= 0 + 0t + 0t^2 + 0t^3$

exactly when $a_1 = a_2 = a_3 = 0$

ie. $\bar{p}(t) = a_0$, a constant polynomial.

6. Consider the map $T : \mathbb{P}_3 \to \mathbb{P}_3$ given by

$$T(\bar{p}) = \bar{p}'$$

where $\bar{p}'$ is the first derivative of the polynomial $\bar{p}$.
(For example, if $\bar{p}(t) = 1 + 3t + 5t^2 + t^3$, then $\bar{p}'(t) = 3 + 10t + 3t^2$.)
Note that T is a linear transformation (you do not need to prove this).
Please find a basis for the kernel of T.

Hint: begin by finding a description of the elements of the kernel of T.

The kernel of T is the set

$\ker T = \{\bar{p} \mid T(\bar{p}) = \bar{0}_{\mathbb{P}_3}\}$.

Note that $\bar{0}_{\mathbb{P}_3} = 0 + 0t + 0t^2 + 0t^3 \quad (= 0)$

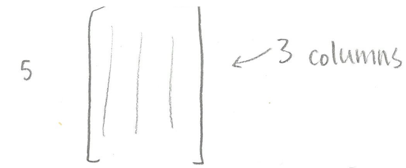
Thus the kernel of T is the set of polynomials whose 1st derivative is 0.

* From calculus, the polynomials whose derivatives are 0 are the constant polynomials $\bar{p}(t) = a_0 + 0t + 0t^2 + 0t^3$.

Thus $\ker T = \{a_0 + 0t + 0t^2 + 0t^3 \mid a_0 \in \mathbb{R}\}$

$= \text{Span}\{1\}$ ← Here, 1 is the constant polynomial $q(t) = 1 + 0t + 0t^2 + 0t^3$

So a basis for $\ker T$ is $\boxed{\{1\}}$.



7. Suppose A is a 5×3 maxtrix. Suppose that for each $\bar{b}$ for which $A\bar{x} = \bar{b}$ has a solution, the solution set to $A\bar{x} = \bar{b}$ is infinite.

(a) What are the possible values for the rank of A? Please justify your response.

Since $A\bar{x} = \bar{0}$ has infinite solutions, we know that A must have a free variable.

Thus, A can have at most 2 pivots; it can have 0, 1, or 2 pivots.

This happens when A is the 0-matrix

rank A = dim(Col A) = # pivots in A.

So $\boxed{\text{rank } A \text{ can be } 0, 1, \text{ or } 2.}$

(b) What are the possible dimensions for Nul A. Please justify your response.

The dimension of Nul A is the dimension of the solution set of $A\bar{x} = \bar{0}$. The dimension of this set equals the number of free variables in A. Since $A\bar{x} = \bar{0}$ has infinite solutions, we know there is at least one free variable in A. A has 3 columns so this means there can be 1, 2, or 3 free variables. Thus dim Nul A = 1, 2, or 3