Math 200 - Linear Algebra Name: Solutions

Final: Practice Exam

Please be sure to neatly **show and explain all of your work** and clearly label your answers. This exam is a closed-book, closed-notebook exam. Calculators are not allowed.

Please write and sign the Honor Pledge here when you are done:

Signed:

Problem	Points
1	/12
2	/12
3	/10
4	/12
5	/12
6	/10
7	/12
Total	/80

Note: This practice test covers only material that we have covered since the second midterm. To review material from earlier in the semester, please use your midterms and practice midterms.

1. Is the following matrix A diagonalizable? If so, give P and D such that $A = PDP^{-1}$. If not, explain why not.

$$A = \begin{bmatrix} 1 & 1 \\ -4 & 5 \end{bmatrix}$$

We start by finding the eigenvalues of A:

$\hookrightarrow \lambda$ an eigenvalue $\Leftrightarrow$ $\det(A - \lambda I) = 0$.

$$\det(A - \lambda I) = \det\begin{bmatrix} 1-\lambda & 1 \\ -4 & 5-\lambda \end{bmatrix} = (1-\lambda)(5-\lambda) + 4$$

$$= 5 - 6\lambda + \lambda^2 + 4$$

$$= \lambda^2 - 6\lambda + 9 = (\lambda - 3)^2.$$

So the eigenvalue is 3, with multiplicity 2. For A to be diagonalizable, the 3-eigenspace must be 2-dim'l. To check, find $\dim \mathrm{Nul}(A - 3I)$.

one-free variable

$$A - 3I = \begin{bmatrix} -2 & 1 \\ -4 & 2 \end{bmatrix} \rightsquigarrow \begin{bmatrix} \boxed{2} & -1 \\ 0 & 0 \end{bmatrix} \Rightarrow \dim \mathrm{Nul}(A - 3I) = 1.$$

Since the 3-eigenspace has dimension 1, not 2, $\boxed{\text{A is not diagonalizable}}$.

2. For each of the following, find the matrix of the linear transformation T relative to the basis $\mathcal{B}$ for the domain and $\mathcal{F}$ for the codomain.

(a) $T : \mathbb{R}^2 \to M_{2\times 2}$ given by

$$T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} 3a+b & -2a+5b \\ -a+2b & a-8b \end{bmatrix}.$$

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}, \quad \mathcal{F} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} = 3\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - 2\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - 1\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + 1\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow \left[T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)\right]_{\mathcal{F}} = \begin{bmatrix} 3 \\ -2 \\ -1 \\ 1 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 5 \\ 2 & -8 \end{bmatrix} = 1\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 5\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 2\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} - 8\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow \left[T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)\right]_{\mathcal{F}} = \begin{bmatrix} 1 \\ 5 \\ 2 \\ -8 \end{bmatrix}$$

Thus $\boxed{{}_{\mathcal{F}}[T]_{\mathcal{B}} = \begin{bmatrix} 3 & 1 \\ -2 & 5 \\ -1 & 2 \\ 1 & -8 \end{bmatrix}}$.

(b) $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by $T(\bar{x}) = \bar{x}$.

$$\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \end{bmatrix} \right\}, \quad \mathcal{F} = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right\}.$$

Note: we can see this by inspection ... or use method (*)

$$T\left(\begin{bmatrix} 2 \\ 4 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 4 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 0\begin{bmatrix} 1 \\ 3 \end{bmatrix} \Rightarrow \left[T\left(\begin{bmatrix} 2 \\ 4 \end{bmatrix}\right)\right]_{\mathcal{F}} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 2 \\ 5 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 5 \end{bmatrix} = 1\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 1\begin{bmatrix} 1 \\ 3 \end{bmatrix} \Rightarrow \left[T\left(\begin{bmatrix} 2 \\ 5 \end{bmatrix}\right)\right]_{\mathcal{F}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

so $\boxed{{}_{\mathcal{F}}[T]_{\mathcal{B}} = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}}$

(*) If you can't see $[T(b_i)]_{\mathcal{F}}$ immediately, can also row reduce, all at once to get each b_i in terms of $\mathcal{F}$:

$$\left[\begin{array}{cc|cc} 1 & 1 & 2 & 2 \\ 2 & 3 & 4 & 5 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 1 & 2 & 2 \\ 0 & 1 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 1 \end{array}\right]$$

$f_1 \quad f_2 \quad T(b_1) \quad T(b_2)$

$[T(b_1)]$ $[T(b_2)]$.

so $\boxed{{}_{\mathcal{F}}[T]_{\mathcal{B}} = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}}$.

3. Please mark the following true or false. If the statement is true, give a short explanation why it is true. If it is false, give a counterexample that shows that it is false.

(a) For a noninvertible $n \times n$ matrix A, the 0-eigenspace of A is the same as Nul A.

$\boxed{\text{True}}$ Since A is not invertible, $A\bar{x} = \bar{0}$ has a nontrivial solution, say $\bar{v}$. This says $A\bar{v} = \bar{0} = 0\bar{v}$, so 0 is an eigenvalue of A. Furthermore, the 0-eigenspace of A is the set of all vectors $\bar{v}$ for which $A\bar{v} = 0\bar{v} = \bar{0}$, ie. all vectors $\bar{v}$ for which $A\bar{v} = \bar{0}$. This is exactly the same as Nul A.

(b) Suppose that $\{\bar{v}_1, \ldots, \bar{v}_p\}$ is a basis for a subspace W of $\mathbb{R}^n$. If $\bar{y} \in \mathbb{R}^n$, then

$$\text{proj}_W(\bar{y}) = \left(\frac{\bar{y} \cdot v_1}{v_1 \cdot v_1}\right)\bar{v}_1 + \cdots + \left(\frac{\bar{y} \cdot v_p}{v_p \cdot v_p}\right)\bar{v}_p.$$

key idea

$\boxed{\text{False}}$ This equation holds when $\{\bar{v}_1, \ldots, \bar{v}_p\}$ is an orthogonal basis, but not necessarily in general.

For example, if $W = \left\{ \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \middle| x, y \in \mathbb{R} \right\} \subset \mathbb{R}^3$, then $\left\{ \underset{\bar{v}_1}{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}, \underset{\bar{v}_2}{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}} \right\}$ is a basis for W. If $y = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, then $\text{proj}_W \bar{y} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ but the formula above gives $\left(\frac{\bar{y} \cdot \bar{v}_1}{\bar{v}_1 \cdot \bar{v}_1}\right)\bar{v}_1 + \left(\frac{\bar{y} \cdot \bar{v}_2}{\bar{v}_2 \cdot \bar{v}_2}\right)\bar{v}_2 = \frac{1}{1}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \neq \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

4. Suppose a basis for a subspace W of $\mathbb{R}^4$ is given by

$$\left\{ \overset{\bar{x}_1}{\begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}}, \overset{\bar{x}_2}{\begin{bmatrix} 1 \\ 6 \\ 0 \\ 0 \end{bmatrix}}, \overset{\bar{x}_3}{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 3 \end{bmatrix}} \right\}.$$

Find an orthogonal basis for W.

↳ we want to use Gram-Schmidt to convert this to an orthogonal basis. $\{\bar{v}_1, \bar{v}_2, \bar{v}_3\}$

let $\bar{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

Then $\text{proj}_{v_1} \bar{x}_2 = \dfrac{\bar{x}_2 \cdot \bar{v}_1}{\bar{v}_1 \cdot \bar{v}_1} \bar{v}_1 = \dfrac{6}{3}\begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 2 \\ 2 \end{bmatrix}.$

So let $\bar{v}_2 = \bar{x}_2 - \text{proj}_{v_1} \bar{x}_2 = \begin{bmatrix} 1 \\ 6 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 2 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ -2 \\ -2 \end{bmatrix}.$

Let $W_2 = \text{span}\{\bar{v}_1, \bar{v}_2\} = \text{span}\left\{\begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ -2 \\ -2 \end{bmatrix}\right\}$

Then $\text{proj}_{W_2} \bar{x}_3 = \dfrac{\bar{x}_3 \cdot \bar{v}_1}{\bar{v}_1 \cdot \bar{v}_1} \bar{v}_1 + \dfrac{\bar{x}_3 \cdot \bar{v}_2}{\bar{v}_2 \cdot \bar{v}_2} \bar{v}_2 = \dfrac{3}{3}\begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \dfrac{-5}{1+16+4+4}\begin{bmatrix} 1 \\ 4 \\ -2 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -1/5 \\ -4/5 \\ 2/5 \\ 2/5 \end{bmatrix} = \begin{bmatrix} -1/5 \\ 1/5 \\ 7/5 \\ 7/5 \end{bmatrix}$

So let $\bar{v}_3 = \bar{x}_3 - \text{proj}_{W_2} \bar{x}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 3 \end{bmatrix} - \begin{bmatrix} -1/5 \\ 1/5 \\ 7/5 \\ 7/5 \end{bmatrix} = \begin{bmatrix} 6/5 \\ -1/5 \\ -7/5 \\ 8/5 \end{bmatrix}$

So our orthogonal basis is:

$$\left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ -2 \\ -2 \end{bmatrix}, \begin{bmatrix} 6/5 \\ -1/5 \\ -7/5 \\ 8/5 \end{bmatrix} \right\}$$

5. Consider the system $A\bar{x} = \bar{b}$ where

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 5 \\ 2 & 0 \end{bmatrix}, \quad \bar{b} = \begin{bmatrix} 21 \\ 1 \\ 2 \end{bmatrix}.$$

Notice that this system is inconsistent. Find all least-squares solutions of the system.

↳ we must solve $A^T A \hat{x} = A^T \bar{b}$ for $\hat{x}$.

$A^T A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 5 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 5 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 5 & 5 \\ 5 & 26 \end{bmatrix}$

$A^T \bar{b} = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 5 & 0 \end{bmatrix}\begin{bmatrix} 21 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 26 \end{bmatrix}$

So Solve: $\begin{bmatrix} 5 & 5 \\ 5 & 26 \end{bmatrix}\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 26 \end{bmatrix}.$

All three methods say: $\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

three methods: ① Row reduce: $\left[\begin{array}{cc|c} 5 & 5 & 5 \\ 5 & 26 & 26 \end{array}\right] \rightsquigarrow \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & 1 \end{array}\right] \rightsquigarrow \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 1 \end{array}\right]$

② Inverse: $\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} = \begin{bmatrix} 5 & 5 \\ 5 & 26 \end{bmatrix}^{-1}\begin{bmatrix} 5 \\ 26 \end{bmatrix}$

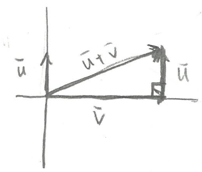
(lucky!)

③ Notice that $\begin{bmatrix} 5 \\ 26 \end{bmatrix}$ is the second column of $\begin{bmatrix} 5 & 5 \\ 5 & 26 \end{bmatrix}$ so $\begin{bmatrix} 5 \\ 26 \end{bmatrix} = \begin{bmatrix} 5 & 5 \\ 5 & 26 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}.$

6. Use dot products to prove the Pythagorean Theorem, which says that for $\bar{u}, \bar{v} \in \mathbb{R}^n$,

$$\|\bar{u} + \bar{v}\|^2 = \|\bar{u}\|^2 + \|\bar{v}\|^2$$

if and only if $\bar{u} \perp \bar{v}$.



first,

defn length

$$\|\bar{u} + \bar{v}\|^2 = (\bar{u} + \bar{v}) \cdot (\bar{u} + \bar{v})$$

linearity of dot product → $= \bar{u}\cdot\bar{u} + \bar{u}\cdot\bar{v} + \bar{v}\cdot\bar{u} + \bar{v}\cdot\bar{v}$

symmetry of dot product → $= \bar{u}\cdot\bar{u} + 2\bar{u}\cdot\bar{v} + \bar{v}\cdot\bar{v}$

defn of length. → $= \|\bar{u}\|^2 + 2\bar{u}\cdot\bar{v} + \|\bar{v}\|^2$

Thus if $\|\bar{u}+\bar{v}\|^2 = \|\bar{u}\|^2 + \|\bar{v}\|^2$, then $2\bar{u}\cdot\bar{v} = 0$, so $\bar{u}\cdot\bar{v} = 0$, so $\bar{u} \perp \bar{v}$.

On the other hand, if $\bar{u} \perp \bar{v}$, then $\bar{u}\cdot\bar{v} = 0$, so $2\bar{u}\cdot\bar{v} = 0$,

So $\|\bar{u}+\bar{v}\|^2 = \|\bar{u}\|^2 + \cancel{2\bar{u}\cdot\bar{v}}^{\,0} + \|\bar{v}\|^2$

$= \|\bar{u}\|^2 + \|\bar{v}\|^2$.

7. The matrix

$$A = \begin{bmatrix} 4 & 0 & 2 \\ 2 & 3 & 4 \\ 0 & 0 & 3 \end{bmatrix}.$$

has eigenvalues 3 and 4. Find a basis for the 3-eigenspace of A. Is A diagonalizable?

We must find a basis for Nul $(A-3I)$, ie. the soln set of $(A-3I)\bar{x} = \bar{0}$.

$$A - 3I = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow[\text{reduce}]{\text{row}} \begin{array}{c} \begin{array}{ccc} & \text{free} & \text{free} \end{array} \\ \begin{bmatrix} \boxed{1} & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array} \Rightarrow \begin{array}{l} x_1 = -2x_3 \\ x_2 = \text{free} \\ x_3 = \text{free} \end{array}$$

Soln set:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

So a basis for the 3-eigenspace is $\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}$

Since the 3-eigenspace is 2-dim'l, and the 4-eigenspace must be at least 1-dim'l, we conclude that the 4-eigenspace is exactly 1-dim'l. b/c A is a 3x3 matrix. Thus we have a full basis of $\mathbb{R}^3$ of eigenvectors, so A is diagonalizable.