Systems of Linear Equations

Ex. $4x + 2y = 8$ is a linear equation.

$\hookrightarrow y = -2x + 4$ ← set of pts. (x, y) satisfying eqn. is a line in $\mathbb{R}^2$.

$3x + y + 2z = 6$ is also a linear equation.

↖ set of pts. (x, y, z) satisfying eqn. is a plane in $\mathbb{R}^3$

General: $a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = b$ is a linear equation.

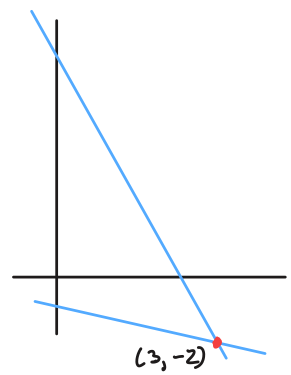
↑ Here, $a_1, a_2, \ldots, a_n$ and b are real numbers, a.k.a. scalars

$x_1, x_2, \ldots, x_n$ are variables (like x, y, z)

Ex $7x_1 + \sqrt{2}\, x_2 - 13x_3 + \frac{\pi}{2} x_4 = -10$

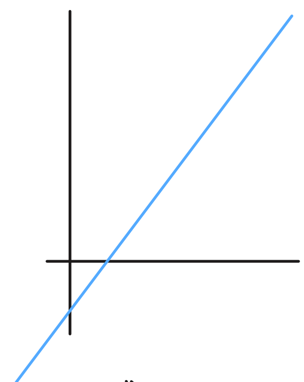
Ex Systems of linear equations:

1. $x + 2y = -1$, $2x + y = 4$ → $(3, -2)$
2. $4x - 2y = 6$, $2x - y = 3$ → $(x, 2x - 3)$
3. $2x + y = 6$, $4x + 2y = 5$ → no soln.



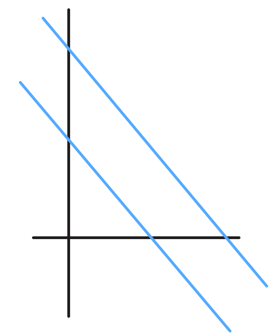
consistent

↑ a unique solution



consistent

↑ infinitely many solutions



inconsistent

↑ no solutions.

Main Q's: Given a system of linear equations:

1. Does a solution exist?

2. If so, is the solution unique?

The solution set is the set of all solutions of the system.

Matrix Notation

Given a system of linear equations:

e.g.
$$\begin{aligned} x_1 - 3x_2 &= 5 \\ -x_1 + x_2 + 5x_3 &= 2 \\ x_2 + x_3 &= 0 \end{aligned}$$

we can write the coefficient matrix

$$\begin{aligned} x_1 - 3x_2 &= 5 \\ -x_1 + x_2 + 5x_3 &= 2 \\ x_2 + x_3 &= 0 \end{aligned}$$

$$\begin{bmatrix} 1 & -3 & 0 \\ -1 & 1 & 5 \\ 0 & 1 & 1 \end{bmatrix}$$

and the augmented matrix of the system.

$$\left[\begin{array}{ccc|c} 1 & -3 & 0 & 5 \\ -1 & 1 & 5 & 2 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

3x4 matrix

In general, a matrix with

n columns

m rows

has size $m \times n$

("m by n")

Solving systems of linear equations

↳ want an algorithm ... will use matrices.

Note:
- interchanging equations
- multiplying an eqn by a nonzero scalar
- replacing eqn (A) by (A) + c(B)

} are all reversible. do not change the solution set of the system.

These operations transform original system to an equivalent system

↑ same solution set

Solving systems of linear equations using matrix notation

$$\begin{aligned} x_1 - 3x_2 \quad &= 5 \\ -x_1 + x_2 + 5x_3 &= 2 \\ x_2 + x_3 &= 0 \end{aligned}$$

write as an augmented matrix

row

$$\begin{bmatrix} 1 & -3 & 0 & 5 \\ -1 & 1 & 5 & 2 \\ 0 & 1 & 1 & 0 \end{bmatrix} \xrightarrow{\text{replace } ② \text{ with } ① + ②} \begin{bmatrix} 1 & -3 & 0 & 5 \\ 0 & -2 & 5 & 7 \\ 0 & 1 & 1 & 0 \end{bmatrix} \xrightarrow{-\frac{1}{2}②} \begin{bmatrix} 1 & -3 & 0 & 5 \\ 0 & 1 & -\frac{5}{2} & -\frac{7}{2} \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

$$\xrightarrow{\text{replace } ③ \text{ with } ③ + -1②} \begin{bmatrix} 1 & -3 & 0 & 5 \\ 0 & 1 & -5/2 & -7/2 \\ 0 & 0 & 7/2 & 7/2 \end{bmatrix} \xrightarrow{\frac{2}{7}③} \begin{bmatrix} 1 & -3 & 0 & 5 \\ 0 & 1 & -5/2 & -7/2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Note: We have transformed the original system into an equivalent system:

$$\begin{aligned} x_1 - 3x_2 \quad &= 5 \\ x_2 - \tfrac{5}{2}x_3 &= -\tfrac{7}{2} \\ x_3 &= 1 \end{aligned}$$

same soln set as the first

It can be solved by back-substitution, but instead we continue with row operations.

$$\begin{bmatrix} 1 & -3 & 0 & 5 \\ 0 & 1 & -\frac{5}{2} & -\frac{7}{2} \\ 0 & 0 & 1 & 1 \end{bmatrix} \xrightarrow{\text{replace } ② \text{ with } \frac{5}{2}③ + ②} \begin{bmatrix} 1 & -3 & 0 & 5 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\xrightarrow{\text{replace } ① \text{ with } ① + 3②} \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

We now have another equivalent system whose solution set is easy to read:

$$\begin{aligned} x_1 \quad\quad\quad &= 2 \\ x_2 \quad\quad &= -1 \\ x_3 &= 1 \end{aligned}$$

(plug into original system to check answer.)

Echelon and reduced echelon form of a matrix:

The matrix

$$\begin{array}{cc} \begin{bmatrix} \boxed{-1} & 10 & -6 & 0 & 1 \\ 0 & \boxed{7} & 2 & 0 & 4 \\ 0 & 0 & 0 & 0 & \boxed{-8} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} & \begin{array}{l} \text{leading entries} \\ \\ \\ \leftarrow \text{zero row} \end{array} \end{array}$$

is in <u>echelon</u> form.

1. All zero rows are at the bottom.
2. Each leading entry is in a column to the right of leading entries above it.
3. Column entries below leading entries are zero.

↳ Matrix has a general "upper triangular" shape to it.

The matrix

all entries other than the leading 1's are 0.

$$\begin{bmatrix} \boxed{1} & 3 & 4 & 0 & 0 & 8 \\ 0 & 0 & 0 & \boxed{1} & 0 & -2 \\ 0 & 0 & 0 & 0 & \boxed{1} & 9 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

is in <u>reduced echelon</u> form.

1. It satisfies the properties of echelon form.
2. Each leading entry in each nonzero row is a 1.
3. Each leading 1 is the only nonzero entry in its column.

"Theorem"

<u>Thm</u> Each matrix can be row reduced to a unique reduced echelon matrix.

<u>Row reduction algorithm</u>: (note similarity w/ solving system of linear equations)

<u>Ex</u> Reduce

$$\begin{bmatrix} 0 & 0 & 1 & 5 & 3 & 23 \\ 3 & 6 & 1 & 2 & 0 & 8 \\ 1 & 2 & 0 & 0 & -1 & 0 \\ -1 & -2 & 2 & 1 & -1 & 1 \end{bmatrix}$$

to echelon and reduced echelon forms.

$$\begin{bmatrix} 0 & 0 & 1 & 5 & 3 & 23 \\ 3 & 6 & 1 & 2 & 0 & 8 \\ 1 & 2 & 0 & 0 & -1 & 0 \\ -1 & -2 & 2 & 1 & -1 & 1 \end{bmatrix}$$

$$\Big\downarrow \ ① \leftrightarrow ③$$

Goal: Since leftmost column is nonzero, make top entry in column nonzero.

$$\begin{bmatrix} 1 & 2 & 0 & 0 & -1 & 0 \\ 3 & 6 & 1 & 2 & 0 & 8 \\ 0 & 0 & 1 & 5 & 3 & 23 \\ -1 & -2 & 2 & 1 & -1 & 1 \end{bmatrix}$$

replace ② with $-3①+②$
and
replace ④ with $①+④$

Goal: eliminate nonzero entries in first column below leading entry.

$$\begin{bmatrix} 1 & 2 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 2 & 3 & 8 \\ 0 & 0 & 1 & 5 & 3 & 23 \\ 0 & 0 & 2 & 1 & -2 & 1 \end{bmatrix}$$

Ignore first row and repeat process with remaining matrix.

replace ③ with $-②+③$
replace ④ with $-2②+④$

Nonzero leading entry in first nonzero column? ✓
Move to eliminate nonzero entries below leading entry.

$$\begin{bmatrix} 1 & 2 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 2 & 3 & 8 \\ 0 & 0 & 0 & 3 & 0 & 15 \\ 0 & 0 & 0 & -3 & 0 & -15 \end{bmatrix}$$

Ignore second row ... move on to third.
Nonzero leading entry in first nonzero column? ✓
Move to eliminate nonzero entries below leading entry.

replace ④ with $③+④$

$$\begin{bmatrix} 1 & 2 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 2 & 3 & 8 \\ 0 & 0 & 0 & 3 & 0 & 15 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

this is in <u>echelon</u> form (nonunique)

... continue to <u>reduced echelon</u> form.

$$\Big\downarrow \ \tfrac{1}{3}③$$

Goal: transform leading entry in ③ to a 1.

$$\begin{bmatrix} 1 & 2 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 2 & 3 & 8 \\ 0 & 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

replace ② with $-2③ + ②$

$$\begin{bmatrix} 1 & 2 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 3 & -2 \\ 0 & 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Goal: eliminate remaining nonzero entries above leading 1 in 4^{th} column.

this is in reduced echelon form (unique).

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echelon form

$$\begin{bmatrix} \boxed{1} & 2 & 0 & 0 & -1 & 0 \\ 0 & 0 & \boxed{1} & 2 & 3 & 8 \\ 0 & 0 & 0 & \boxed{3} & 0 & 15 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

reduced echelon form

$$\begin{bmatrix} \boxed{1} & 2 & 0 & 0 & -1 & 0 \\ 0 & 0 & \boxed{1} & 0 & 3 & -2 \\ 0 & 0 & 0 & \boxed{1} & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Columns 1, 3, and 4 are pivot columns. Pivots/pivot positions are boxed.

Note: if a reduced augmented matrix has no

$$\begin{bmatrix} 0 & \cdots & 0 & b \end{bmatrix}$$

$b \neq 0$

rows, it is consistent.

says $0x_1 + 0x_2 + \cdots + 0x_n = b$,

i.e. $0 = b$

Ex Sps. augmented matrix has reduced echelon form

$$\begin{bmatrix} \boxed{1} & -7 & 0 & 6 & 5 \\ 0 & 0 & \boxed{1} & -2 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} x_1 - 7x_2 \quad\quad + 6x_4 &= 5 \\ x_3 - 2x_4 &= 3 \end{aligned}$$

x_1 and x_3 (pivot columns) are basic variables.

x_2 and x_4 are free variables.