Note: if a reduced augmented matrix has no

$$\begin{bmatrix} 0 & \cdots & 0 & b \end{bmatrix} \quad \leftarrow b \neq 0$$

rows, it is consistent.

says $0x_1 + 0x_2 + \dots + 0x_n = b$, i.e. $0 = b$ ✗

Ex Sps. augmented matrix has reduced echelon form

$$\begin{bmatrix} \boxed{1} & -7 & 0 & 6 & 5 \\ 0 & 0 & \boxed{1} & -2 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x_1 - 7x_2 \quad + 6x_4 = 5$$
$$x_3 - 2x_4 = 3$$

x_1 and x_3 (pivot columns) are basic variables.

x_2 and x_4 are free variables

General solution can be expressed

$$x_1 = 5 + 7x_2 - 6x_4$$
$$x_2 = \text{free}$$
$$x_3 = -3 \quad + 2x_4$$
$$x_4 = \text{free}$$

choose x_2, x_4 anything... gives infinite # of solns.

x_1, x_3 depend on x_2, x_4.

* Conclusion: for a consistent matrix, free variables ⇒ infinitely many solutions.

Vectors in $\mathbb{R}^n$

Linear algebra is the study of vector spaces. Vector spaces are made up of vectors. $\mathbb{R}^n$ is an example of a vector space.

Algebraic interpretation

A vector in $\mathbb{R}^n$ is a list of n numbers, in a column vector:

column vector ⇝ $n \times 1$ matrix

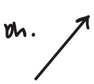
$$\bar{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \quad \leftarrow \text{each } u_i \in \mathbb{R}$$

"is an element of"

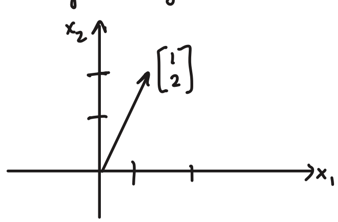
Ex: $\begin{bmatrix} 1 \\ -7 \\ 3 \end{bmatrix} \in \mathbb{R}^3$

Geometric interpretation (in $\mathbb{R}^2$)

A vector is an object with a magnitude and direction.

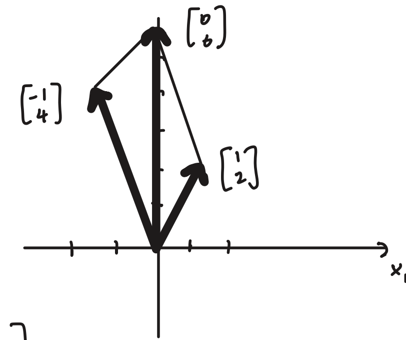


Place tail at origin to give it coordinates.



Vectors can be added:

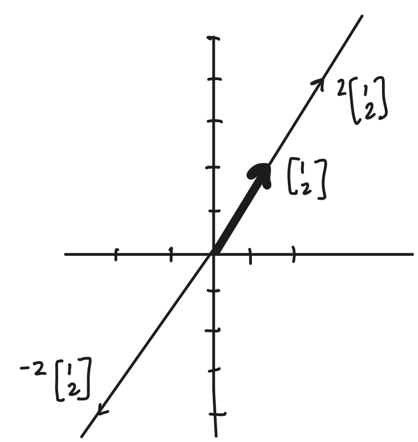
Ex $\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \end{bmatrix}$



general: $\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ \vdots \\ u_n + v_n \end{bmatrix}$

and multiplied by scalars:

Ex. $2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$



general: $c \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} cu_1 \\ cu_2 \\ \vdots \\ cu_n \end{bmatrix}$

For all vector spaces, operations on vectors are addn. and scalar mult.

All vector spaces have a zero vector, $\bar{0}$.

In $\mathbb{R}^n$, it is: $\bar{0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

See: algebraic properties of $\mathbb{R}^n$ on p. 29

say: commutativity, associativity, distribution, etc.

Compare: defn of vector space on p. 202-3

Linear combinations and spans

Sps. $\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_k$ are vectors in $\mathbb{R}^n$ and $c_1, c_2, \ldots, c_k$ are scalars.

$$\bar{y} = c_1\bar{v}_1 + c_2\bar{v}_2 + \cdots + c_k\bar{v}_k$$

is a linear combination of $\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_k$ with weights $c_1, c_2, \ldots, c_k$.

Ex $$3\begin{bmatrix} 1 \\ 5 \end{bmatrix} - 1\begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 15 \end{bmatrix}$$

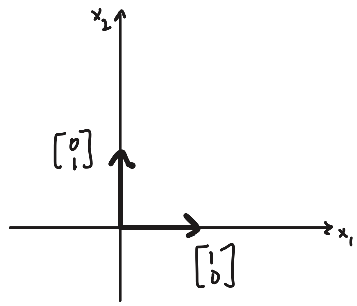
weights

linear comb.

The span of $\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_k$ is the set of all linear combinations of $\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_k$.

<u>Ex</u> $\text{span}\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\} = \mathbb{R}^2$

(explain)



<u>Ex</u> Is $\bar{b} = \begin{bmatrix}3\\0\end{bmatrix}$ in $\text{span}\left\{\begin{bmatrix}1\\-1\end{bmatrix}, \begin{bmatrix}4\\2\end{bmatrix}\right\}$?

"such that"

i.e. are there numbers x_1 and x_2 s.t.

$$x_1\begin{bmatrix}1\\-1\end{bmatrix} + x_2\begin{bmatrix}4\\2\end{bmatrix} = \begin{bmatrix}3\\0\end{bmatrix} ?$$

i.e. is there a solution to

$$\begin{aligned} x_1 + 4x_2 &= 3 \\ -x_1 + 2x_2 &= 0 \ ? \end{aligned}$$

$$\begin{bmatrix}1 & 4 & 3\\ -1 & 2 & 0\end{bmatrix} \xrightarrow{\text{row reduce}} \begin{bmatrix}1 & 0 & 1\\ 0 & 1 & 1/2\end{bmatrix}$$

consistent, so <u>yes</u>:

$$1\begin{bmatrix}1\\-1\end{bmatrix} + \frac{1}{2}\begin{bmatrix}4\\2\end{bmatrix} = \begin{bmatrix}3\\0\end{bmatrix}$$