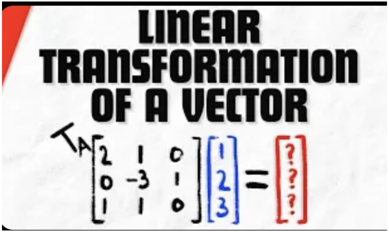
MATH 200C: Linear Algebra



Class 10: March 2, 2026

- Notes on Assignment 8
- Matrix of a Linear Transformation

Announcements

Exam 1:Wednesday, 7 PM -
No Time Limit

Last Name A - M : Warner 100
Last Name N - Z : Warner 105

No Books,Computers, Smart Phones, etc.

One Page of Your Own Notes OK

Linear Transformations

We call T a **matrix transformation** if $T(\mathbf{x}) = A\mathbf{x}$ for some matrix A.

Definition: A transformation T from $\mathbb{R}^n$ to $\mathbb{R}^m$ is called a **linear transformation** if for all $\mathbf{u}$ and $\mathbf{v}$ in the domain of T and all scalars c, d both

$$T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}),$$

and

$$T(c\mathbf{u}) = cT(\mathbf{u})$$

Note: Every matrix transformation is a linear transformation but Not all transformations are linear

Theorem: If T is a linear transformation, then

$$T(\mathbf{0}) = \mathbf{0} \text{ and}$$

$$T(c\mathbf{u} + d\mathbf{v}) = cT(\mathbf{u}) + dT(\mathbf{v})$$

Proof:

$$T(c\mathbf{u} + d\mathbf{v}) = T(c\mathbf{u}) + T(d\mathbf{v}) = cT(\mathbf{u}) + dT(\mathbf{v})$$

Theorem: if A is an m $\times$ n matrix, then the transformation $T(\mathbf{x}) = A\mathbf{x}$ is a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$.

BIG QUESTION: If T is a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$, must T be of the form $T(\mathbf{x}) = A\mathbf{x}$ for some $m \times n$ matrix?
If so, how do we find A?

Theorem 10: If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, then there exists a unique matrix A such that $T(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x}$ in $\mathbb{R}^n$.

A is the $m \times n$ matrix whose jth column is $T(\mathbf{e}_j)$ where $\mathbf{e}_j$ is the jth column of the identity matrix in $\mathbb{R}^n$. We call A the **standard matrix** for T.

Standard Unit Vectors

In $\mathbb{R}^2$: $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

In $\mathbb{R}^3$: $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

In $\mathbb{R}^4$: $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \mathbf{e}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ The set $\{\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_n\}$ is linearly independent and spans $\mathbb{R}^n$.

Example

Let T be a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ and suppose $\mathbf{x}$ is an arbitrary vector in $\mathbb{R}^2$. Then

$$T(\mathbf{x}) = T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = T\left(x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = x_1 T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) + x_2 T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$$

Now suppose $T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} a \\ b \end{bmatrix}$ and $T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} c \\ d \end{bmatrix}$
Then

$$T(\mathbf{x}) = x_1 \begin{bmatrix} a \\ b \end{bmatrix} + x_2 \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \mathbf{x} = A\mathbf{x}$$

Simple Two Step Procedure

(1) Find the images of the standard basis vectors $\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_n$ for $\mathbb{R}^n$

(2) Use the images obtained in Step 1 as the successive columns of the matrix A

Example: Find the standard matrix for the linear Transformation

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_3 - 4x_4 \\ 2x_1 + x_2 \\ x_2 - x_3 + x_4 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 & 1 & -4 \\ 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 \end{bmatrix}$$

Definitions: A mapping T from $\mathbb{R}^n$ to $\mathbb{R}^m$ is **onto (surjective)** means for each $\mathbf{b}$ in $\mathbb{R}^m$, there is **at least one** $\mathbf{x}$ in $\mathbb{R}^n$ such that $T(\mathbf{x}) = \mathbf{b}$.

A mapping T from $\mathbb{R}^n$ to $\mathbb{R}^m$ is **one-to-one (injective)** one-to-one ((injective) means for each $\mathbf{b}$ in $\mathbb{R}^m$ there is **at most one** $\mathbf{x}$ in $\mathbb{R}^n$ such that $T(\mathbf{x}) = \mathbf{b}$

Theorem 11: Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then T is one-to-one if and only if the equation $T(\mathbf{x}) = \mathbf{0}$ has only the trivial solution.

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Theorem 12: Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation with A as its standard matrix. Then

- T maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ if and only if the columns of A span $\mathbb{R}^m$
- T is one-to-one if and only if the columns of A are a linearly independent set.