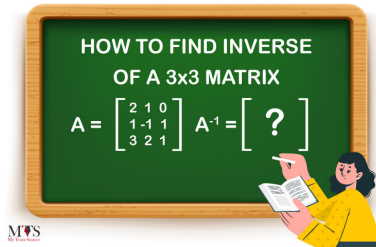


MATH 200C: Linear Algebra



Class 13: March 9, 2026



- ▶ Notes on Assignment 11
- ▶ Inverse of a Matrix (Last Time)
- ▶ Additional Matrix Operations in MATLAB
- ▶ Project 1: Native Matrix Multiplication



Friday's and Next Monday's Class on Zoom

Matrix Operations

Scalar Multiplication

Ingredients: Any Scalar r and Any Matrix A

Results: $(rA)_{ij} = rA_{ij}$

Idea: Multiply Every Entry in A by the Scalar r

Example:

$$3 \begin{bmatrix} 1 & -2 & 4 \\ 5 & 7 & 0 \end{bmatrix} = \begin{bmatrix} (3)(1) & (3)(-2) & (3)(4) \\ (3)(5) & (3)(7) & (3)(0) \end{bmatrix} = \begin{bmatrix} 3 & -6 & 12 \\ 15 & 21 & 0 \end{bmatrix}$$

Matrix Operations

Matrix Addition

Ingredients: Two Matrices A and B **OF THE SAME SIZE**

$$\text{Results: } (A + B)_{ij} = A_{ij} + B_{ij}$$

Idea: Add Together Corresponding Entries in Two Matrices

$$\text{Example: } \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} + \begin{bmatrix} 7 & 8 \\ 4 & 2 \\ 0 & 6 \end{bmatrix} = \begin{bmatrix} 1+7 & 2+8 \\ 3+4 & 4+2 \\ 5+0 & 6+6 \end{bmatrix} = \begin{bmatrix} 8 & 10 \\ 7 & 6 \\ 5 & 12 \end{bmatrix}$$

Matrix Multiplication

Definition: If A is an $m \times n$ matrix and B is an $n \times p$ matrix with columns $\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_p$, then the product AB is the $m \times p$ matrix $A\mathbf{b}_1, A\mathbf{b}_2, \dots, A\mathbf{b}_p$; That is,

$$AB = A[\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_p] = [A\mathbf{b}_1, A\mathbf{b}_2, \dots, A\mathbf{b}_p]$$

Each column of AB is a linear combination of the columns of

A with weights from the corresponding columns of B

Relation To Composition of Linear Transformations

If S is a linear transformation from \mathbb{R}^p to \mathbb{R}^n with standard matrix B and T is a linear transformation from \mathbb{R}^n to \mathbb{R}^m with standard matrix A , then composition given by $(TS)(x) = T(S(x))$ is a linear transformation from \mathbb{R}^p to \mathbb{R}^m with standard matrix AB .

Row-Column Rule for Computing AB

If the product AB is defined, then the entry in row i and column j of AB is the sum of the products of the corresponding entries from row i of matrix A and column j of matrix B ; that is,

$$(A)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j}\dots + a_{in}b_{nj} = \sum_{k=1}^n a_{ik}b_{kj}$$

$$(\mathbf{AB})_{ij} = (\text{row } i \text{ of } \mathbf{A}) (\text{column } j \text{ of } \mathbf{B})$$

Example of Matrix Multiplication

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 3 & 2 & 0 \\ 3 & 9 & 2 & 2 & 6 \\ 5 & 1 & 8 & 4 & 1 \end{bmatrix}, B = \begin{bmatrix} 2 & 4 & 6 \\ 1 & 3 & 5 \\ 0 & 3 & 6 \\ 4 & 5 & 6 \\ 1 & 1 & 1 \end{bmatrix}$$

$$\begin{aligned} (AB)_{3,2} &= [3 \quad 9 \quad 2 \quad 2 \quad 6] \begin{bmatrix} 4 \\ 3 \\ 3 \\ 5 \\ 1 \end{bmatrix} \\ &= (3)(4) + (9)(3) + (2)(3) + (2)(5) + (6)(1) \\ &= 12 + 27 + 6 + 10 + 6 = 61 \end{aligned}$$

Transpose of a Matrix

Definition: The **transpose** of an $m \times n$ matrix, denoted A^T , is the $n \times m$ whose columns are formed from the corresponding rows of A .

The entry in row i and column j of A^T is the entry in column j and row i of A .

Example: If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$, then $A^T = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$

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Theorem 3: Let A and B denote matrices whose sizes are appropriate for the following sums and products

- ▶ $(A^T)^T = A$
- ▶ $(A + B)^T = A^T + B^T$
- ▶ $(rA)^T = r A^T$ for any scalar r
- ▶ $(AB)^T = B^T A^T$

transpose of a product is product of transpose in reverse order

Inverse of a Matrix

Definition: A square $n \times n$ matrix A is **invertible (nonsingular)** if there is an $n \times n$ matrix C such that AC and CA are the identity

$n \times n$ matrix:

$$AC = CA = I = I_n$$

and C is called an **inverse** of A .

Note: There is at most one inverse for a given square matrix A

. Proof: If B and C are each inverses of A ,
then $B = BI = B(AC) = BA(C) = IC = C$.

If A is invertible, the unique inverse of A is denoted A^{-1} .

A non-invertible matrix is also called a **singular matrix**.

Examples of invertible and noninvertible matrices:

The Inverse of $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ is $A^{-1} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}$

The Matrix $B = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ has no inverse.

Theorem 4: Inverses of 2×2 matrices If $ad - bc \neq 0$, then the matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible with $A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$
If $ad - bc = 0$, then A is not invertible.

Theorem 5: If A is an $n \times n$ invertible matrix, then for each \mathbf{b} in \mathbb{R}^n , the equation $A\mathbf{x} = \mathbf{b}$ has the unique solution $\mathbf{x} = A^{-1}\mathbf{b}$.

Theorem 6: If A is invertible, then so A^{-1} and $(A^{-1})^{-1} = A$.
(The inverse of the inverse is the original matrix)

If A and B are invertible, then so is AB with $(AB)^{-1} = B^{-1}A^{-1}$
(the inverse of the product is the product of the inverses in reverse order)

If A is invertible, then so is A^T with $(A^T)^{-1} = (A^{-1})^T$ (The inverse of the transpose is the transpose of the inverse).

Elementary Matrices

Definition: An **elementary matrix** is a matrix obtained by performing a single elementary row operation on an identity matrix.

If an elementary row operation is performed on an $m \times n$ matrix A , we can write the resulting matrix as EA where the $m \times m$ matrix E is the result of applying the same row operation on I_m .

Each elementary matrix E is invertible. The inverse of E is the elementary matrix of the same type that transforms E back into I .

Theorem 7: An $n \times n$ matrix A is invertible if and only if A is row equivalent to I_n .

Any sequence of elementary row operations that reduces A to the Identity matrix also transforms the Identity matrix into the inverse of A .

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An Algorithm for Finding A^{-1}

Row Reduce the Augmented Matrix $[A|I]$.

If A is row equivalent to I , then

$[A|I]$ is row equivalent to $[I|A^{-1}]$.

Otherwise, A does not have an inverse.

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Example: Find the inverse of $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

Form $(A|I) = \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array} \right]$ and reduce to row echelon form

Row 2 = Row 2 - 3* Row 1: $\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -2 & -3 & 1 \end{array} \right]$

Row 2 = $(-1/2)$ Row 2: $\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & 3/2 & -1/2 \end{array} \right]$

Row 1 = Row $-(3/2)$ Row 2: $\left[\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & 1 & 3/2 & -1/2 \end{array} \right]$