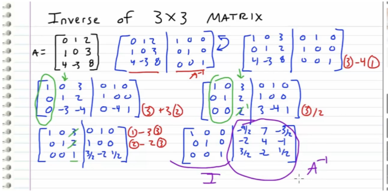
MATH 200C: Linear Algebra



Class 14: March 11, 2026

- Notes on Assignment 12
- Characterizing Invertible Matrices

Friday's and Next Monday's Class on Zoom

- Submitting Homework
- Class Notes Online
- Solutions Online

Inverse of a Matrix

Definition: A square $n \times n$ matrix A is **invertible (nonsingular)** if there is an $n \times n$ matrix C such that AC and CA are the identity $n \times n$matrix:

$$AC = CA = I = I_n$$

and C is called an **inverse** of A.

Note: There is at most one inverse for a given square matrix A . Proof: If B and C are each inverses of A,

then $B = BI = B(AC) = BA(C) = IC = C$.

If A is invertible, the unique inverse of A is denoted A^{-1} .

A non-invertible matrix is also called a **singular matrix** .

Examples of invertible and noninvertible matrices:

The Inverse of $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ is $A^{-1} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}$

The Matrix $B = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ has no inverse.

Theorem 4: Inverses of 2×2 matrices If $ad - bc \neq 0$, then the matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible with $A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$
If $ad - bc = 0$, then A is not invertible.

Theorem 5: If A is an $n \times n$ invertible matrix, then for each $\mathbf{b}$ in $\mathbb{R}^n$, the equation $A\mathbf{x} = \mathbf{b}$ has the unique solution $\mathbf{x} = A^{-1}\mathbf{b}$.

Theorem 6: If A is invertible, then so A^{-1} and $(A^{-1})^{-1} = A$. (The inverse of the inverse is the original matrix)

If A and B are invertible, then so is AB with $(AB)^{-1} = B^{-1}A^{-1}$ (the inverse of the product is the product of the inverses in reverse order)

If A is invertible, then so is A^T with $(A^T)^{-1} = (A^{-1})^T$ (The inverse of the transpose is the transpose of the inverse).

Elementary Matrices

Definition: An **elementary matrix** is a matrix obtained by performing a single elementary row operation on an identity matrix.

If an elementary row operation is performed on an $m \times n$ matrix A, we can write the resulting matrix as EA where the $m \times m$ matrix E is the result of applying the same row operation on I_m .

Each elementary matrix E is invertible. The inverse of E is the elementary matrix of the same type that transforms E back into I.

Theorem 7: An $n \times n$ matrix A is invertible if and only if A is row equivalent to I_n.
Any sequence of elementary row operations that reduces A to the Identity matrix also transforms the Identity matrix into the inverse of A.

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An Algorithm for Finding A^{-1}

Row Reduce the Augmented Matrix $[A|I]$.

If A is row equivalent to I, then

$[A|I]$ is row equivalent to $\left[I|A^{-1}\right]$.

Otherwise, A does not have an inverse.

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Example: Find the inverse of $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

Form $(A|I) = \begin{bmatrix} 1 & 2 & | & 1 & 0 \\ 3 & 4 & | & 0 & 1 \end{bmatrix}$ and reduce to row echelon form

Row 2 = Row 2 - 3* Row 1: $\begin{bmatrix} 1 & 2 & | & 1 & 0 \\ 0 & -2 & | & -3 & 1 \end{bmatrix}$

Row 2 = (-1/2) Row 2: $\begin{bmatrix} 1 & 2 & | & 1 & 0 \\ 0 & 1 & | & 3/2 & -1/2 \end{bmatrix}$

Row 1 = Row -(3/2) Row 2: $\begin{bmatrix} 1 & 0 & | & \mathbf{-2} & \mathbf{1} \\ 0 & 1 & | & \mathbf{3/2} & \mathbf{-1/2} \end{bmatrix}$

Theorem (The Invertible Matrix Theorem): Let A be a square $n \times n$ matrix. Then the following statements are equivalent; that is, for a given matrix A, the statements are either all true or all false.

- A is an invertible matrix.
- A is row equivalent to the $n \times n$ identity matrix.
- A has n pivot positions.
- The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
- The columns of A form a linearly independent set.
- The linear transformation $\mathbf{x} \to A$x is one – to – one.
- The equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for all $\mathbf{b}$ in $\mathbb{R}^n$
- The columns of A span $\mathbb{R}^n$.
- The linear transformation $\mathbf{x} \to A\mathbf{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$.
- There is an $n \times n$ matrix C such that $CA = I$.
- There is an $n \times n$ matrix D such that $AD = I$.
- A^T is an invertible matrix.

Theorem 9: Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation and let A be the standard matrix for T. Then T is invertible if and only if A is an invertible matrix.

In that case, the linear transformation S given by $S(\mathbf{x}) = A^{-1}\mathbf{x}$ is the unique transformation satisfying $S(T(\mathbf{x}) = \mathbf{x}$ and $T(S(\mathbf{x}))) = \mathbf{x})$ for all $\mathbf{x}$ in $\mathbb{R}^n$.