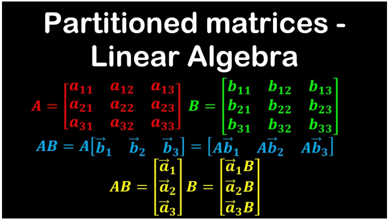
MATH 200C: Linear Algebra



Class 15: Friday, March 13, 2026

- Notes on Assignment 12
- Partitioned Matrix

Friday's and Next Monday's Class on Zoom

Partitioned Matrices

A Partitioned Matrix is a partitioned matrix is a division of a matrix into smaller rectangular matrices called **submatrices** or **blocks**.

$$A = \begin{bmatrix} 1 & -3 & 0 & 4 & 2 & -4 \\ -1 & 10 & 5 & -2 & 0 & 3 \\ -5 & 6 & 2 & 4 & -5 & -7 \end{bmatrix}$$

$$A = \left[\begin{array}{cc|cccc} 1 & -3 & 0 & 4 & 2 & -4 \\ \hline -1 & 10 & 5 & -2 & 0 & 3 \\ -5 & 6 & 2 & 4 & -5 & -7 \end{array}\right]$$

Example

$$A = \left[\begin{array}{cc|cccc} 1 & -3 & 0 & 4 & 2 & -4 \\ \hline -1 & 10 & 5 & -2 & 0 & 3 \\ -5 & 6 & 2 & 4 & -5 & -7 \end{array}\right]$$

with submatrices $\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$

$$A_{11} = \begin{bmatrix} 1 & -3 \end{bmatrix} \qquad A_{12} = \begin{bmatrix} 0 & 4 & 2 & -4 \end{bmatrix}$$

$$A_{21} = \begin{bmatrix} -1 & 10 \\ -5 & 6 \end{bmatrix} \qquad \begin{bmatrix} 5 & -2 & 0 & 3 \\ 2 & 4 & -5 & -7 \end{bmatrix}$$

Can Partition a Matrix in Various Ways

$$A = \begin{bmatrix} 1 & -3 & 0 & 4 & 2 & -4 \\ -1 & 10 & 5 & -2 & 0 & 3 \\ -5 & 6 & 2 & 4 & -5 & -7 \end{bmatrix}$$

$$A = \left[\begin{array}{ccc|cc|c} 1 & -3 & 0 & 4 & 2 & -4 \\ -1 & 10 & 5 & -2 & 0 & 3 \\ \hline -5 & 6 & 2 & 4 & -5 & -7 \end{array}\right]$$

with submatrices $\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \end{bmatrix}$

$$\begin{array}{lll} A_{11} : 2 \times 3 & A_{12} : 2 \times 2 & A_{13} : 2 \times 1 \\ A_{21} : 1 \times 3 & A_{22} : 1 \times 2 & A_{23} : 1 \times 1 \end{array}$$

Notation We typically use capital letters to represent each block or submatrix with subscripts indicating its location

$$A = \left[\begin{array}{ccc|cc|c} 1 & -3 & 0 & 4 & 2 & -4 \\ -1 & 10 & 5 & -2 & 0 & 3 \\ \hline -5 & 6 & 2 & \mathbf{4} & \mathbf{-5} & -7 \end{array}\right]$$

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & \mathbf{A_{22}} & A_{23} \end{bmatrix}$$

Partitioning Helps Us See Structure of a Matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

$$A = \left[\begin{array}{ccc|cc|c} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ \hline 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$A = \left[\begin{array}{ccc|cc|c} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ \hline 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$A = \begin{bmatrix} I_{3\times 3} & 0_{3\times 2} & B_{3\times 1} \\ 0_{2\times 3} & I_{2\times 2} & 0_{2\times 1} \\ B^T_{1\times 3} & 0_{1\times 2} & 1 \end{bmatrix}$$

Operations with Partitioned Matrices
Scalar Multiplication and Addition

If matrices are the same size and are partitioned in precisely the same way, then we can multiply by scalars and add matrices in the same way as with ordinary matrices, except each block of $A+B$ is a matrix sum.

If $A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \end{bmatrix}$ and $B = \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \end{bmatrix}$,

then for any scalars c and d we have

$$A + B = \begin{bmatrix} cA_{11} + dB_{11} & cA_{12} + dB_{12} & cA_{13} + dB_{13} \\ cA_{21} + dB_{21} & cA_{22} + dB_{22} & cA_{23} + dB_{23} \end{bmatrix}$$

Operations with Partitioned Matrices
Partitioned Matrix Multiplication

If the dimensions of matrices A and B match the required operation such that $(m \times n)(n \times p) = (m \times p)$, then we say the partitions of A and B are conformable for block multiplication. Then

$$\text{If } A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \\ A_{31} & A_{32} \end{bmatrix} \text{ and}$$

$$B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}, \text{ then}$$

$$AB = \underbrace{\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \\ A_{31} & A_{32} \end{bmatrix}}_{3\times 2} \underbrace{\begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}}_{2\times 2}$$

$$= \underbrace{\begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \\ A_{31}B_{11} + A_{32}B_{21} & A_{31}B_{12} + A_{32}B_{22} \end{bmatrix}}_{3\times 2}$$

Partitioning can be immensely helpful in the sense that we can now perform powers and operations with similar block matrices.

For example, suppose the matrices below are partitioned for block multiplication.

$$\begin{bmatrix} I & 0 \\ A & I \end{bmatrix} \begin{bmatrix} A & B \\ -C & 0 \end{bmatrix}$$

The Product is

$$\begin{bmatrix} I(A) + 0(-C) & I(B) + 0(0) \\ A(A) + I(-C) & A(B) + I(0) \end{bmatrix} = \begin{bmatrix} A & B \\ A^2 - C & AB \end{bmatrix}$$

$$\begin{bmatrix} I \times A + 0 \times (-C) & I \times B + 0 \times 0 \\ A \times A + I \times (-C) & A \times B + I \times 0 \end{bmatrix} = \begin{bmatrix} A & B \\ A^2 - C & AB \end{bmatrix}$$

Column – Row Expansion of AB

If A is an $m \times n$ and B is $n \times p$, then

$$AB = \begin{bmatrix} \text{column}_1(A) & \text{column}_2(A) & \ldots \text{column}_n(A) \end{bmatrix} \begin{bmatrix} row_1(B) \\ row_2(B) \\ \vdots \\ row_n(B) \end{bmatrix}$$

$$= \text{column}_1(A)\text{row}_1(B) + \text{column}_2(A)\text{row}_2(B) + \cdots + \text{column}_n(A)\text{row}_n(B)$$

where column_k is the kth column of A
and $\text{row}_k(\text{B})$ is the kth row of B

Operations with Partitioned Matrices
Block Diagonal Matrices and Their Inverses

A partitioned matrix A is **block diagonal** if the matrices on the main diagonal are square and all other position matrices are zero.

$$A = \begin{bmatrix} D_1 & 0 & \dots & 0 \\ 0 & D_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & D_k \end{bmatrix}$$

Each of $D_1, D_2, \dots, D_k$ is a square matrix but they can be of different sizes.

The matrix A is invertible if and only if every D_i is invertible.

$$A^{-1} = \begin{bmatrix} D_1^{-1} & 0 & \dots & 0 \\ 0 & D_2^{-1} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & D_k^{-1} \end{bmatrix}$$

Example of Inverse of Partitioned Matrix

$$A = \left[\begin{array}{cc|cc|c} 1 & -1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 \\ \hline 0 & 0 & 5 & 4 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 2 \end{array}\right]$$

$$A^{-1} = \left[\begin{array}{cc|cc|c} 2/3 & 1/3 & 0 & 0 & 0 \\ -1/3 & 2/3 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & -4 & 0 \\ 0 & 0 & -1 & 5 & 0 \\ \hline 0 & 0 & 0 & 0 & 1/2 \end{array}\right]$$

$$A = \begin{array}{c} B \\ C \\ D \end{array} \begin{bmatrix} 1 & -3 & 0 & 4 & 2 & -4 \\ -1 & 10 & 5 & -2 & 0 & 3 \\ -5 & 6 & 2 & 4 & -5 & -7 \end{bmatrix}$$

$$A = \begin{array}{c} \\ B \\ C \\ D \\ E \end{array} \begin{array}{c} \begin{array}{cccc} B & C & D & E \end{array} \\ \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline a & b & c & d \\ e & f & g & h \end{array}\right] \end{array}$$