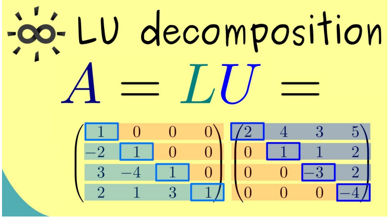
MATH 200C: Linear Algebra



Class 16: Monday, March 16, 2026

- Notes on Assignment 13
- Matrix Factorizations

Regular In-Person Classes Resume on Wednesday

Factorization

A factorization of a number is an expression of that number as a product of two or more numbers

Example: $30 = (2)(3)(5)$

A factorization of a matrix is an expression of that matrix as a product of two or more matrices:

$$A = BC$$

$$\begin{bmatrix} 8 & 5 \\ 20 & 13 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}$$

LU Factorization

Definitions:

- An **upper triangular matrix** is one whose entries **below** the main diagonal are all 0's.
  Any non-zero entry lies on or above the main diagonal.
- An **lower triangular matrix** is one whose entries **above** the main diagonal are all 0's.
  Any non-zero entry lies on or below the main diagonal.

Definition: If A is an $m \times n$ matrix, then an **LU Factorization** of A is a representation of A as a product LU of a (lower triangular matrix L) and an (upper triangular matrix U)

$$A = LU$$

where L is a square $m \times m$ lower triangular matrix with 1's on the main diagonal and U is an $m \times n$ echelon form of A.

$$\begin{bmatrix} 1 & -2 & 1 & 2 & 3 \\ 3 & -5 & 7 & 7 & 13 \\ 2 & 0 & 18 & 11 & 23 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 1 & 2 & 3 \\ 0 & 1 & 4 & 1 & 4 \\ 0 & 0 & 0 & 3 & 1 \end{bmatrix}$$

Why Do an LU Factorization?

Major Goal: Solve Systems of Linear Equations $A\mathbf{x} = \mathbf{b}$
If $A = LU$, then $A\mathbf{x} = \mathbf{b}$ becomes $LU\mathbf{x} = \mathbf{b}$ or $L(U\mathbf{x}) = \mathbf{b}$

Writing $\mathbf{y} = U\mathbf{x}$, we can solve $A\mathbf{x} = \mathbf{b}$ by solving the pair of equations

$$\boxed{\begin{aligned} L\mathbf{y} &= \mathbf{b} \\ U\mathbf{x} &= \mathbf{y} \end{aligned}}$$

First, Solve $L\mathbf{y} = \mathbf{b}$ for $\mathbf{y}$

Then, Solve $U\mathbf{x} = \mathbf{y}$ for $\mathbf{x}$
Each system is easy to solve because of its triangular nature and it involves fewer numerical operations.

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ * & 1 & 0 & 0 \\ * & * & 1 & 0 \\ * & * & * & 1 \end{bmatrix} \begin{bmatrix} \blacksquare & * & * & * & * \\ 0 & \blacksquare & * & * & * \\ 0 & 0 & 0 & \blacksquare & * \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\text{Solving } \begin{bmatrix} 1 & 0 & 0 & 0 \\ * & 1 & 0 & 0 \\ * & * & 1 & 0 \\ * & * & * & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

$$\text{Solving } \begin{bmatrix} \blacksquare & * & * & * & * \\ 0 & \blacksquare & * & * & * \\ 0 & 0 & 0 & \blacksquare & * \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$$

Suppose A can be reduced to an echelon form U using only row replacements that add a multiple of one row to another row below it.

Algorithm for an LU Factorization

Step 1: Reduce A to an echelon form U by a sequence of row replacement operations, if possible.

Step 2: Place entries in L such that the same sequence of row operations reduces L to I.

Example: Find an LU factorization of $\begin{bmatrix} 1 & 4 & -1 & 5 \\ 3 & 7 & -2 & 9 \\ -2 & -3 & 1 & -4 \\ -1 & 6 & -1 & 7 \end{bmatrix}$

$$A = \begin{bmatrix} \textcircled{1} & 4 & -1 & 5 \\ 3 & 7 & -2 & 9 \\ -2 & -3 & 1 & -4 \\ -1 & 6 & -1 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & -1 & 5 \\ 0 & \textcircled{-5} & 1 & -6 \\ 0 & 5 & -1 & 6 \\ 0 & 10 & -2 & 12 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & -1 & 5 \\ 0 & -5 & 1 & -6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = U$$

$$\begin{bmatrix} \textcircled{1} \\ 3 \\ -2 \\ -1 \end{bmatrix} \quad \begin{bmatrix} \textcircled{-5} \\ 5 \\ 10 \end{bmatrix}$$

Use the last two columns of I_4 to make L unit lower triangular.

$\div 1$ $\quad \div -5$

$$\begin{bmatrix} 1 & & & \\ 3 & 1 & & \\ -2 & -1 & 1 & \\ -1 & -2 & 0 & 1 \end{bmatrix}, \quad L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ -2 & -1 & 1 & 0 \\ -1 & -2 & 0 & 1 \end{bmatrix}$$

Alan Turing (1948)
British Mathematician
1912 – 1954

Tadeusz Banachiewicz (1938)
Polish Astronomer
1882 – 1954

- Structural Engineering: Analyzes forces in buildings and bridges to ensure safety and stability.
- Electrical Engineering: Simulates circuit behavior over time, including frequency sweep analysis and solving for voltages and currents.
- Computer Graphics & Animation: Transforms 3D objects, renders lighting, and simulates physical movements like cloth draping.
- Computational Fluid Dynamics (CFD): Solves Navier-Stokes equations to simulate fluid flow.
- Machine Learning & Data Analysis: Used for fitting regression models, calculating matrix determinants, and solving large-scale linear systems.
- Robotics: Solves kinematic equations for real-time robotic movement.
- Finance & Economics: Optimizes resources and forecasts market trends.

Paraphrased Note From Text: In practical work, row interchanges are nearly always needed, because partial pivoting is used for high accuracy. (Recall that this procedure selects, among the possible choices for a pivot, an entry in the column having the largest absolute value.)

To handle row interchanges, modify the LU factorization to produce an L that is permuted lower triangular: a rearrangement of the rows of L can make L(unit) lower triangular.

The resulting permuted LU factorization solves $A\mathbf{x} = \mathbf{b}$ as before, except that the reduction of $[L\,\mathbf{b}]$ to $[I\;\mathbf{y}]$ follows the order of the pivots in L from left to right, starting with the pivot in the first column.

References to "LU factorization" usually include the possibility that L might be permuted lower triangular.