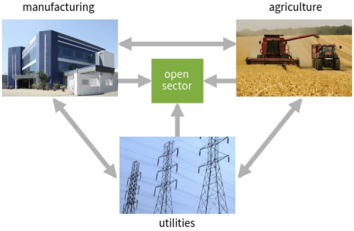
# MATH 200C: Linear Algebra




Class 17: Wednesday, March 18, 2026

Handy Handouts

- Notes on Assignment 15
- LU Factorization Example
- Leontief Input – Output Model

# Leontief Input – Output Model

Ingredients

- Economy has $n$ sectors producing goods and services
- **Production vector** $\mathbf{x}$ in $\mathbb{R}^n$: Yearly output of each sector
- Open sector: Consumers goods and services but does not produce them.
- **Final Demand Vector** $\mathbf{d}$: values of goods and services demanded from various sectors by nonproductive sector.
- **Intermediate Demand**: What productive sectors need as inputs for their own production.
- For each sector, there is a **unit consumption vector** in $\mathbb{R}^n$ listing inputs needed per unit of output of that sector. These vectors form the **Connsumption Matrix** $C$.
- Inputs and Outputs measured in dollars.

## Major Question of Leontief Model

Is there a production vector $\mathbf{x}$ that satisfies intermediate demand and final demand $\mathbf{d}$ together?

$$\mathbf{x} = C\mathbf{x} + \mathbf{d}$$

$$\mathbf{x} = C\mathbf{x} + \mathbf{d}$$

$$I\mathbf{x} = C\mathbf{x} + \mathbf{d}$$

$$I\mathbf{x} - C\mathbf{x} = \mathbf{d}$$

$$(I - C)\mathbf{x} = \mathbf{d}$$

$$\mathbf{x} = (I - C)^{-1}\mathbf{d}$$

**Theorem 11**: Let $C$ be the consumption matrix for an economy and let $\mathbf{d}$ be the final demand.

If $C$ and $\mathbf{d}$ have nonnegative entries and if each column sum of $C$ is less than 1, then $(I - C)$ is invertible and the production vector

$$\mathbf{x} = (I - C)^{-1}\mathbf{d}$$

has nonnegative entries and is the unique solution of

$$\mathbf{x} = C\mathbf{x} + \mathbf{d}$$

Bruce Peterson and Michael Olinick, "Leontief Models, Markov Chains, Substochastic Matrices, and Positive Solutions of Matrix Equations," *Mathematical Modelling*, Volume 3, Issue 3, 1982, 221 – 239.

Read The Paper

Suppose the industries receive the demand $\mathbf{d}$ .
They set their production levels at $\mathbf{x} = \mathbf{d}$, to exactly meet the final demand $\mathbf{d}$.
As they prepare to produce $\mathbf{d}$, they send out orders for their raw materials and other inputs.
This creates an intermediate demand of $C\mathbf{d}$ for inputs.
To meet the additional demand of $C\mathbf{d}$ the industries need additional inputs of $C(C\mathbf{d}) = C^2\mathbf{d}$
This creates a second round of intermediate demand.
When the industries decide to produce even more to meet this new demand, they create a third round of demand, namely $C(C^2\mathbf{d}) = C^3\mathbf{d}$

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# A Formula for $(\mathbf{I} - \mathbf{C})^{-1}$

| | Demand that must be met | Inputs needed to meet this demand |
|---|---|---|
| Final Demand | $\mathbf{d}$ | $C\mathbf{d}$ |
| Intermediate Demand | | |
| 1st Round | $C\mathbf{d}$ | $C(C\mathbf{d}) = C^2\mathbf{d}$ |
| 2nd Round | $C^2\mathbf{d}$ | $C(C^2\mathbf{d}) = C^3\mathbf{d}$ |
| 3rd Round | $C^3\mathbf{d}$ | $C(C^3\mathbf{d}) = C^4\mathbf{d}$ |
| 4th Round | $C^4\mathbf{d}$ | $C(C^4\mathbf{d}) = C^5\mathbf{d}$ |

Production level $\mathbf{x}$ must meet all of these demands

$$\mathbf{x} = \mathbf{d} + C\mathbf{d} + C^2\mathbf{d} + C^3\mathbf{d} + C^4\mathbf{d} + \dots$$

$$\mathbf{x} = (I + C + C^2 + C^3 + C^4 + \dots)\mathbf{d}$$

$$(I - C)^{-1} \approx I + C + C^2 + C^3 + C^4 + \cdots + C^n$$

if the column sums of $C$ are all strictly less than 1

Example

$$C = \begin{bmatrix} 0.3000 & 0.3000 & 0.1000 & 0.1000 \\ 0.4000 & 0.2000 & 0.4000 & 0.1000 \\ 0.1000 & 0 & 0.3000 & 0.1000 \\ 0.1000 & 0.0500 & 0 & 0.2000 \end{bmatrix}$$

$$(I - C)^{-1} = \begin{bmatrix} 2.0572 & 0.7996 & 0.7508 & 0.4510 \\ 1.2413 & 1.7480 & 1.1762 & 0.5207 \\ 0.3417 & 0.1441 & 1.5597 & 0.2557 \\ 0.3347 & 0.2092 & 0.1674 & 1.3389 \end{bmatrix}$$

$$\text{With } \mathbf{d} = \begin{bmatrix} 40 \\ 60 \\ 80 \\ 100 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} 235.4254 \\ 300.6974 \\ 172.6639 \\ 173.2218 \end{bmatrix}$$

## The Economic Relevance of the Entries of $(I - C)^{-1}$

The entries tell how the production vector **x** must change when the final demand **d** changes.

Entries in column $j$ are the increased amounts the various sectors will have to produce to satisfy a unit increase in the final demand for output from sector $j$.

$$(I - C)^{-1} = \begin{bmatrix} 2.0572 & 0.7996 & 0.7508 & 0.4510 \\ 1.2413 & 1.7480 & 1.1762 & 0.5207 \\ 0.3417 & 0.1441 & 1.5597 & 0.2557 \\ 0.3347 & 0.2092 & 0.1674 & 1.3389 \end{bmatrix}$$

Suppose demand from third sector increases from current demand of 80 to 81.

$$\text{With } \mathbf{d}_{old} = \begin{bmatrix} 40 \\ 60 \\ \mathbf{80} \\ 100 \end{bmatrix}, \mathbf{x}_{old} = \begin{bmatrix} 235.4254 \\ 300.6974 \\ 172.6639 \\ 173.2218 \end{bmatrix}$$

$$\text{Now } \mathbf{d}_{new} = \begin{bmatrix} 40 \\ 60 \\ \mathbf{81} \\ 100 \end{bmatrix}, \mathbf{x}_{new} = \begin{bmatrix} 236.1762 \\ 301.8735 \\ 174.2236 \\ 173.3891 \end{bmatrix}$$

Approximation: Using $(I - C)^{-1} \approx I + C + C^2 + C^2 + \cdots + C^{12}$,

$$\text{we obtain } \mathbf{x} \approx \begin{bmatrix} 231.0973 \\ 295.8384 \\ 171.2470 \\ 171.8722 \end{bmatrix} \text{ instead of } \mathbf{x} = \begin{bmatrix} 235.4254 \\ 300.6974 \\ 172.6639 \\ 173.2218 \end{bmatrix}$$