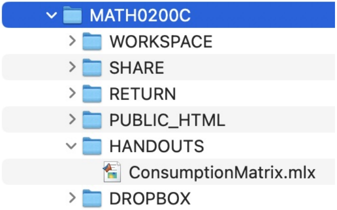
MATH 200C: Linear Algebra

Class 18: Friday, March 20, 2026

- Notes on Assignment 16
- Determinants

Exam 2: Wednesday, April 8

Assignment 18 (Due Wednesday, April 1):
Exercise 13 of Section 2.6
Features 7×7 Consumption Matrix



Leontief Input – Output Model

- Production Vector $\mathbf{x}$
- External Demand Vector $\mathbf{d}$
- For each sector, there is a **unit consumption vector** in $\mathbb{R}^n$ listing inputs needed per unit of output of that sector. These vectors form the **Connsumption Matrix** C.

Inputs and Outputs measured in dollars.

Problem: Find $\mathbf{x}$ so that $\mathbf{x} = \mathrm{C}\ \mathbf{x} + \mathbf{d}$

Solution: $\mathbf{x} = (I - C)^{-1}\mathbf{d}$

The Economic Relevance of the Entries of $(I - C)^{-1}$

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The entries tell how the production vector **x** must change when the final demand **d** changes.

Entries in column j are the increased amounts the various sectors will have to produce to satisfy a unit increase in the final demand for output from sector j.

$$(I - C)^{-1} = \begin{bmatrix} 2.0572 & 0.7996 & 0.7508 & 0.4510 \\ 1.2413 & 1.7480 & 1.1762 & 0.5207 \\ 0.3417 & 0.1441 & 1.5597 & 0.2557 \\ 0.3347 & 0.2092 & 0.1674 & 1.3389 \end{bmatrix}$$

Suppose demand from third sector increases from current demand of 80 to 81.

$$\text{With } \mathbf{d}_{old} = \begin{bmatrix} 40 \\ 60 \\ \color{blue}{\mathbf{80}} \\ 100 \end{bmatrix}, \mathbf{x}_{old} = \begin{bmatrix} 235.4254 \\ 300.6974 \\ 172.6639 \\ 173.2218 \end{bmatrix}$$

$$\text{Now } \mathbf{d}_{new} = \begin{bmatrix} 40 \\ 60 \\ \color{blue}{\mathbf{81}} \\ 100 \end{bmatrix}, \mathbf{x}_{new} = \begin{bmatrix} 236.1762 \\ 301.8735 \\ 174.2236 \\ 173.3891 \end{bmatrix} \quad \mathbf{x}_{new} - \mathbf{x}_{old} = \begin{bmatrix} .7508 \\ 1.1762 \\ 1.5597 \\ 0.1674 \end{bmatrix}$$

$$(I - C)^{-1} = \begin{bmatrix} 2.0572 & 0.7996 & \color{red}{\mathbf{0.7508}} & 0.4510 \\ 1.2413 & 1.7480 & \color{red}{\mathbf{1.1762}} & 0.5207 \\ 0.3417 & 0.1441 & \color{red}{\mathbf{1.5597}} & 0.2557 \\ 0.3347 & 0.2092 & \color{red}{\mathbf{0.1674}} & 1.3389 \end{bmatrix}$$

Determinants

Introduction To Determinants

Key Fact: The **determinant** of a square matrix is a number which is nonzero if and only if the matrix is invertible.

For a 1×1 matrix $A = [\, a_{11}]$, the determinant of A is the number a_{11}.

For a 2×2 matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, the determinant of A (denoted det A) is defined as the number $a_{11}a_{22} - a_{21}a_{12}$. Easier to remember as det $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$.

Notation: For a square matrix A, we let A_{ij} denote the submatrix of A obtained by deleting the ith row and jth column of A.

Definition: For $n \geq 2$, the **determinant of an $n \times n$ matrix** $A = [a_{ij}]$ is the sum of n terms of the form $\pm a_{1j}\, det\, A_{1j}$ with plus and minus signs alternating, where the entries $a_{11}, a_{12}, \ldots, a_{1n}$ are from the first row of A:

$$\det A = a_{11}\, detA_{11} - a_{12}\, detA_{12} + \cdots + (-1)^{n+1} a_{1n}\, detA_{1n} = \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det A_{1j}$$

Definition: The **(i, j)-cofactor** of A is the number $C_{ij} = (-1)^{i+j} \det A_{ij}$.

With this notation, we have $\det A = a_{11}C_{11} + a_{12}C_{12} + \ldots + a_{1n} C_{1n}$ which is called the **cofactor expansion across the first row** of A

Theorem 1: The determinant of any $n \times n$ matrix A can be computed by a cofactor expansion across any row or down any column. The expansion across the ith row using the cofactors is

$$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + a_{i3}C_{i3} + \cdots + a_{in}C_{n1}$$

The cofactor expansion down the jth column is

$$\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + a_{3j}C_{3j} + \cdots + a_{nj}C_{nj}$$

The plus or minus sign in the (i,j)-cofactor depends on the position of the number a_{ij} in the matrix, not on the *sign* of the number a_{ij}. The factor $(-1)^{i+j}$ creates this pattern of signs:

$$\begin{bmatrix} + & - & + & \cdots \\ - & + & - & \\ + & - & + & \\ \vdots & & & \ddots \end{bmatrix}$$

Theorem 2: The determinant of a triangular matrix is the product of the diagonal element

Properties of Determinants

Theorem 3: Determinants and the Elementary Row Operations: Let A be square matrix

(a) If we add a multiple of one row of A to another row to produce a matrix B, then $\det B = \det A$.
(b) If we interchange two rows of A to produce a matrix B, then $\det B = - \det A$.
(c) If we multiply one row of A by a constant k to produce a matrix B, then $\det B = k \det A$.

Theorem 4: A square matrix is invertible if and only if its determinant is nonzero. ($\det A \neq 0$)

Theorem 5: The determinant of the transpose of a square matrix equals the determinant of the original matrix

$$(\det A^T = \det A).$$

Theorem 6: Determinant of a product is the product of the determinants: If A and B are $n \times n$ matrices,

$$\text{then } \det(AB) = (\det A)(\det B)$$

Determinants as Area or Volume (Read Pages 191 – 195 of Text)

Theorem 9: If A is a 2×2 matrix, the area of the parallelogram determined by the columns of A is $|\det A|$.

If A is a 3×3 matrix, the volume of the parallelepiped determined by the columns of A is $|\det A|$.

Theorem 10: Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation determined by a 2×2 matrix A. If S is a parallelogram in $\mathbb{R}^2$, then Area of $T(S) = |\det A|$ (area of S).

Let $T: \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation determined by a 3×3 matrix A. If S is a parallelepiped in $\mathbb{R}^3$, then Volume of $T(S) = |\det A|$ (volume of S).


Enjoy
your
Spring
Break!