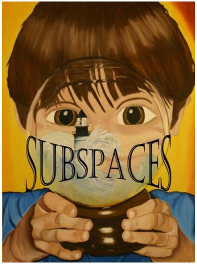
MATH 200C: Linear Algebra



Class 20: Wednesday, April 1, 2026

Notes on Assignment 18

Vector Spaces

Null Spaces, Column Spaces, Row Spaces, and Linear Transformations

Exam 2: Wednesday, April 8

Vector Spaces

Definition: A **vector space** is a nonempty set V of objects, called vectors, on which are defined two operations, called **addition** , denoted $\oplus$, and **multiplication by scalars** (real numbers), denoted $\odot$, subject to the ten axioms (or rules) listed below. The axioms must hold for all vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in V and for all scalars c and d.

1. $\mathbf{u} \oplus \mathbf{v}$ is in $\mathbf{v}$
2. $\mathbf{u} \oplus \mathbf{v} = \mathbf{v} \oplus \mathbf{u}$.
3. $(\mathbf{u} \oplus \mathbf{v}) \oplus \mathrm{w} = \mathbf{u} \oplus (\mathbf{u} \oplus \mathrm{w})$
4. There is a zero vector 0 in V such that $\mathbf{u} \oplus 0 = \mathbf{u}$
5. For each $\mathbf{u}$ in V, there is a vector -$\mathbf{u}$ in V such that $\mathbf{u} \oplus -\mathbf{u} = 0$
6. $c \odot \mathbf{u}$ is in $\mathbf{v}$
7. $c \odot (\mathbf{u} \oplus \mathbf{v}) = \mathrm{c} \odot \mathbf{u} \oplus \mathrm{c} \odot \mathbf{v}$
8. $(\ \mathrm{c} + \mathrm{d}\ ) \odot \mathbf{u} = \mathrm{c} \odot \mathbf{u} \oplus \mathrm{d} \odot \mathbf{u}$
9. $\mathrm{c} \odot (\mathrm{d} \odot \mathbf{u}) = (\mathrm{c} \odot \mathrm{d})\ \mathbf{u}$
10. $1 \odot \mathbf{u} = \mathbf{u}$

Some Examples of Vector Spaces

- $\mathbb{R}^n$, for $n \geq 1$
- All 3×5 matrices
- All $m \times n$ matrices for a fixed m and n.
- All arrows in 3 dimensional space
- All infinite sequences of real numbers $a_1, a_2, a_3, \ldots$
- $\mathbb{P}_n$, polynomials of degree $\leq n$
- $\mathbb{P}$, polynomials
- Real-Valued Differentiable functions on an interval I.

Here's an example of a "weird" vector space

The addition and scalar multiplication operations don't have to look like "normal" operations as long as they obey the rules for vector space operations.

Suppose our set V consists of all ordered pairs (p, q) of real numbers with $p + q = 1$.

Define addition and scalar multiplication as follows:

$(p, q) \oplus (r, s) = (p + r – 1, q + s)$ and $c \odot (p, q) = (cp – c + 1, cq)$.

The set V with these operations turns out to be a vector space, though its operations look nothing like any other "normal" operations.

A few hints to start: the zero vector is (1, 0), and the negative of any vector (p, q) is $(2 – p, –q)$.

Another Weird Example of Vector Spaces

V is the set of all positive numbers with

$$\mathbf{u} \oplus \mathbf{v} = \mathbf{uv}$$
$$c \odot \mathbf{u} = \mathbf{u}^c$$

Examples:

$$3 \oplus 4 = (3)(4) = 12$$
$$3 \odot 4 = 4^3 = 64$$

Yet Another Weird Vector Space

Elements: ordered pairs (a, b) of real numbers (Points in the Plane)

Operations: $(a_1, b_1) \oplus (a_2, b_2) = (a_1 + a_2 + 1, b_1 + b_2 - 2)$
$c \odot (a, b) = (a + c - 1, cb - 2c + 2)$

Subspaces

In many problems, a vector space consists of an appropriate subset of vectors from some larger vector space.

In this case, only three of the ten vector space axioms need to be checked; the rest are automatically satisfied.

Definition: A **subspace** of a vector space V is a subset H of V that has three properties:

(a) The zero vector of V is in H

(b) H is closed under vector addition. That is, for each $\mathbf{u}$ and $\mathbf{v}$ in H , the sum $\mathbf{u} + \mathbf{v}$ is in H

(c) H is closed under multiplication by scalars. That is, for each $\mathbf{u}$ in H and each scalar c, the vector $c\mathbf{u}$ is in H .

Subspaces are closed under addition and scalar multiplication.
Thus they are closed under **Linear Combinations**
We can talk about **Linear Indepedence** and **Span**

Theorem 1:

If $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_p}$ are in a vector space V,
then span $(\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_p})$ is a subspace of V.

Vector Spaces

Null Spaces, Column Spaces, Row Spaces, and Linear Transformations

Definition: The **null space** of an $m \times n$ matrix A, written as $Nul\ A$, is the set of all solutions of the homogeneous equation $A\mathbf{x} = \mathbf{0}$

$$Nul\ A = \{\mathbf{x} : \mathbf{x} \text{ is in } \mathbb{R}^n \text{ and } A\mathbf{x} = \mathbf{0}$$

Theorem 1: The null space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^n$.

Equivalently, the set of all solutions to a system $A\mathbf{x} = \mathbf{0}$ of m homogenous equations in n unknowns is a subspace of $\mathbb{R}^n$.

Definition: The **column space** of an $m \times n$ matrix A, written as $Col\ A$, is the set of all linear combinations of the columns of A. If $A = [\mathbf{a_1} \ldots \mathbf{a_n}]$, then $Col\ A =$ Span $\{\mathbf{a_1} \ldots \mathbf{a_n}\}$.

Theorem 2: The column space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^m$.

$$Col\ A = \{\mathbf{b} : \mathbf{b} = A\mathbf{x} \text{ for some } \mathbf{x} \text{ in } \mathbb{R}^n\}$$

The column space of an $m \times n$ matrix A is all of $\mathbb{R}^m$ if and only if the equation $A\mathbf{x} = \mathbf{b}$ has a solution for all $\mathbf{b}$ in $\mathbb{R}^m$.

Definition: The **row space** of an $m \times n$ matrix A is the set of all linear combinations of the rows of A.

Theorem: The row space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^n$ and is equal to the column space of the transpose of A.

Definition: A **linear transformation T from a vector space V into a vector space W** is a rule that assigns to each vector $\mathbf{x}$ in V a unique vector $T(\mathbf{x})$ in W such that

$T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all $\mathbf{u}, \mathbf{v}$ in V

and

$T(c\mathbf{u}) = cT(\mathbf{u})$ for all u in V and all scalars c.