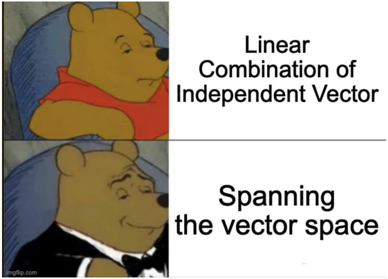
MATH 200C: Linear Algebra



Class 21: Friday, April 3, 2026

- Notes on Assignment 19
- Linearly Independent Sets and Bases

Exam 2: Wednesday, April 8

7 PM – ?

No Calculators, Computers, Phones, Smart Watches, ...

BUT One Sheet of Notes

Vector Spaces

Null Spaces, Column Spaces, Row Spaces, and Linear Transformations (Review)

Definition: The **null space** of an $m \times n$ matrix A, written as Nul A, is the set of all solutions of the homogeneous equation $A\mathbf{x} = \mathbf{0}$

$$Nul\ A = \{\mathbf{x} : \mathbf{x} \text{ is in } \mathbb{R}^n \text{ and } A\mathbf{x} = \mathbf{0}$$

Vector Spaces

Null Spaces, Column Spaces, Row Spaces, and Linear Transformations (Review)

Definition: The **null space** of an $m \times n$ matrix A, written as Nul A, is the set of all solutions of the homogeneous equation $A\mathbf{x} = \mathbf{0}$

$$Nul\ A = \{\mathbf{x} : \mathbf{x} \text{ is in } \mathbb{R}^n \text{ and } A\mathbf{x} = \mathbf{0}$$

Theorem 1: The null space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^n$.

Equivalently, the set of all solutions to a system $A\mathbf{x} = \mathbf{0}$ of m homogenous equations in n unknowns is a subspace of $\mathbb{R}^n$.

Definition: The **column space** of an $m \times n$ matrix A, written as Col A, is the set of all linear combinations of the columns of A.

If $A = [\mathbf{a_1} \dots \mathbf{a_n}]$, then Col A = Span $\{\mathbf{a_1} \dots \mathbf{a_n}\}$.

Definition: The **column space** of an $m \times n$ matrix A, written as Col A, is the set of all linear combinations of the columns of A. If $A = [\mathbf{a_1} \dots \mathbf{a_n}]$, then Col A = Span $\{\mathbf{a_1} \dots \mathbf{a_n}\}$.

Theorem 2: The column space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^m$.

The column space of an $m \times n$ matrix A is all of $\mathbb{R}^m$ if and only if the equation $A\mathbf{x} = \mathbf{b}$ has a solution for all $\mathbf{b}$ in $\mathbb{R}^m$.

Definition: The **row space** of an $m \times n$ matrix A is the set of all linear combinations of the rows of A.

Definition: The **row space** of an $m \times n$ matrix A is the set of all linear combinations of the rows of A.

Theorem: The row space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^n$ and is equal to the column space of the transpose of A.

Old Definition: [Section 1.8] A **linear transformation T from $\mathbb{R}^n$ into $\mathbb{R}^m$** is a rule that assigns to each vector $\mathbf{x}$ in $\mathbb{R}^n$ a unique vector $T(\mathbf{x})$ in $\mathbb{R}^m$ such that

$T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all $\mathbf{u}, \mathbf{v}$ in $\mathbb{R}^n$ and

$T(c\mathbf{u}) = cT(\mathbf{u})$ for all $\mathbf{u}$ in V and all scalars c.

Old Definition: [Section 1.8] A **linear transformation T from $\mathbb{R}^n$ into $\mathbb{R}^m$** is a rule that assigns to each vector $\mathbf{x}$ in $\mathbb{R}^n$ a unique vector $T(\mathbf{x})$ in $\mathbb{R}^m$ such that

$T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all $\mathbf{u}, \mathbf{v}$ in $\mathbb{R}^n$ and

$T(c\mathbf{u}) = cT(\mathbf{u})$ for all $\mathbf{u}$ in V and all scalars c.

Principal Example : **Matrix Transformations**

Let A be any $m \times n$ matrix. Then $T(\mathbf{u}) = A\mathbf{u}$ for $\mathbf{u}$ in $\mathbb{R}^n$.

Old Definition: [Section 1.8] A **linear transformation T from $\mathbb{R}^n$ into $\mathbb{R}^m$** is a rule that assigns to each vector $\mathbf{x}$ in $\mathbb{R}^n$ a unique vector $T(\mathbf{x})$ in $\mathbb{R}^m$ such that

$T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all $\mathbf{u}, \mathbf{v}$ in $\mathbb{R}^n$ and

$T(c\mathbf{u}) = cT(\mathbf{u})$ for all $\mathbf{u}$ in V and all scalars c.

Principal Example : **Matrix Transformations**

Let A be any $m \times n$ matrix. Then $T(\mathbf{u}) = A\mathbf{u}$ for $\mathbf{u}$ in $\mathbb{R}^n$.

Specific Example $A = \begin{pmatrix} 1 & 2 & -3 \\ 5 & -1 & 7 \end{pmatrix}, \mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}.$

$$T(\mathbf{u}) = \begin{pmatrix} 1u_1 + 2u_2 - 3u_3 \\ 5u_1 - 1u_2 + 7u_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \end{pmatrix} u_1 + \begin{pmatrix} 2 \\ -1 \end{pmatrix} u_2 + \begin{pmatrix} -3 \\ 7 \end{pmatrix} u_3$$

Old Definition: [Section 1.8] A **linear transformation T from $\mathbb{R}^n$ into $\mathbb{R}^m$** is a rule that assigns to each vector $\mathbf{x}$ in $\mathbb{R}^n$ a unique vector $T(\mathbf{x})$ in $\mathbb{R}^m$ such that

$$T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}) \text{ for all } \mathbf{u}, \mathbf{v} \text{ in } \mathbb{R}^n$$

and

$$T(c\mathbf{u}) = cT(\mathbf{u}) \text{ for all } \mathbf{u} \text{ in } V \text{ and all scalars } c.$$

Old Definition: [Section 1.8] A **linear transformation T from $\mathbb{R}^n$ into $\mathbb{R}^m$** is a rule that assigns to each vector $\mathbf{x}$ in $\mathbb{R}^n$ a unique vector $T(\mathbf{x})$ in $\mathbb{R}^m$ such that

$$T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}) \text{ for all } \mathbf{u}, \mathbf{v} \text{ in } \mathbb{R}^n$$

and

$$T(c\mathbf{u}) = cT(\mathbf{u}) \text{ for all } \mathbf{u} \text{ in } V \text{ and all scalars } c.$$

Definition: A **linear transformation T from a vector space V into a vector space W** is a rule that assigns to each vector $\mathbf{x}$ in V a unique vector $T(\mathbf{x})$ in W such that

$$T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}) \text{ for all } \mathbf{u}, \mathbf{v} \text{ in } V$$

and

$$T(c\mathbf{u}) = cT(\mathbf{u}) \text{ for all } \mathbf{u} \text{ in } V \text{ and all scalars } c.$$

Examples of Linear Transformations

- Matrix Transformations from $\mathbb{R}^n$ into $\mathbb{R}^m$

Examples of Linear Transformations

- Matrix Transformations from $\mathbb{R}^n$ into $\mathbb{R}^m$
- $T : \mathbb{R}^3 \to \mathbb{P}_5$ by $T \begin{pmatrix} a \\ b \\ c \end{pmatrix} = a + bt^2 + ct^4$.

Examples of Linear Transformations

- Matrix Transformations from $\mathbb{R}^n$ into $\mathbb{R}^m$
- $T : \mathbb{R}^3 \to \mathbb{P}_5$ by $T \begin{pmatrix} a \\ b \\ c \end{pmatrix} = a + bt^2 + ct^4$.
- Calculus Example: V is set of differentiable functions on [-10,10] and W is set of functions on [-10,10] with $T(f) = f'$.

Examples of Linear Transformations

- Matrix Transformations from $\mathbb{R}^n$ into $\mathbb{R}^m$
- $T : \mathbb{R}^3 \to \mathbb{P}_5$ by $T \begin{pmatrix} a \\ b \\ c \end{pmatrix} = a + bt^2 + ct^4$.
- Calculus Example: V is set of differentiable functions on [-10,10] and W is set of functions on [-10,10] with $T(f) = f'$.
- Calculus: Let V be set of continuous functions on [0,10] and $W = V$ with $T(f)$ defined as the antiderivative G of f with $G(0) = 0$.
  Note $T(\cos x) = \sin x$ while $T(\sin x) = -\cos x + 1$.

Examples of Linear Transformations

- Matrix Transformations from $\mathbb{R}^n$ into $\mathbb{R}^m$
- $T : \mathbb{R}^3 \to \mathbb{P}_5$ by $T \begin{pmatrix} a \\ b \\ c \end{pmatrix} = a + bt^2 + ct^4$.
- Calculus Example: V is set of differentiable functions on [-10,10] and W is set of functions on [-10,10] with $T(f) = f'$.
- Calculus: Let V be set of continuous functions on [0,10] and $W = V$ with $T(f)$ defined as the antiderivative G of f with $G(0) = 0$.
  Note $T(\cos x) = \sin x$ while $T(\sin x) = -\cos x + 1$.
- Let V be vector space of all 3×5 matrices and W the vector space of all 2×7 matrices. Define $T : V \to W$ by the following:
  Let $M = \begin{pmatrix} 4 & 3 & 2026 \\ 7 & 4 & 1776 \end{pmatrix}$ and $T(A) = [MA|O]$ where O is the 2×2 zero matrix.

Vector Spaces

Linearly Independent Sets and Bases

Definition: An indexed set of vectors $\{\mathbf{v_1}, \ldots, \mathbf{v_p}\}$ in a vector space V is **linearly independent** if the equation

$$\textbf{(1)} \qquad c_1 v_1 + c_2 v_2 + \cdots + c_p v_p = 0$$

has only the trivial solution, $c_1 = 0, \ldots, c_p = 0$

Vector Spaces

Linearly Independent Sets and Bases

Definition: An indexed set of vectors $\{\mathbf{v_1}, \dots, \mathbf{v_p}\}$ in a vector space V is **linearly independent** if the equation

$$\textbf{(1)} \qquad c_1 v_1 + c_2 v_2 + \cdots + c_p v_p = 0$$

has only the trivial solution, $c_1 = 0, \dots, c_p = 0$

The set $\{\mathbf{v_1}, \dots, \mathbf{v_p}\}$ is said to be **linearly dependent** if **(1)** has a nontrivial solution, that is, if there are some weights, $c_1, \dots, c_p$, not all zero, such that (1) holds.

In such a case, **(1)** is called a **linear dependence relation** among $\mathbf{v_1}, \dots, \mathbf{v_p}$

Theorem 4: An indexed set $\{\mathbf{v_1}, \ldots, \mathbf{v_p}\}$ of two or more vectors, with $\mathbf{v_1} \neq 0$, is linearly dependent if and only if some $\mathbf{v_j}$ (with $j > 1$) is a linear combination of the preceding vectors, $\mathbf{v_1}, \ldots, \mathbf{v_{j-1}}$

Definition Let H be a subspace of a vector space V. A set of vectors $\mathcal{B}$ in V is a **basis** for H if

(i) $\mathcal{B}$ is a linearly independent set, and

(ii) the subspace spanned by $\mathcal{B}$ coincides with H;

that is, $H = \text{Span } \mathcal{B}$.

Theorem 5: The Spanning Set Theorem: Let $S = \{\mathbf{v_1} \dots, \mathbf{v_p}\}$ be a set in a vector space V, and let $H = \text{Span } \{\mathbf{v_1}, \dots, \mathbf{v_p}\}$. If one of the vectors in S — say, v_k —is a linear combination of the remaining vectors in S, then the set formed from S by removing $\mathbf{v_k}$ still spans H.

If $H \neq 0$, some subset of S is a basis for H.

Theorem 6: The pivot columns of a matrix A form a basis for $Col\ A$.

Theorem 7: If two matrices A and B are row equivalent, then their row spaces are the same.

If B is in echelon form, the nonzero rows of B form a basis for the row space of A as well as for that of B.

If f and g are two ordinary functions, how do you add them?

$$(f+g)(x) = f(x) + g(x)$$

How do you multiply a function by a scalar c?

$$(cf)(x) = c \times f(x)$$

Let V and W be a fixed pair of vector. spaces.

Let $\mathbb{T}$ be the set of all linear transformations from V to W

Suppose S and T are members of $\mathbb{T}$ Then

$$(S+T)(\mathbf{u}+\mathbf{v}) = S(\mathbf{u}+\mathbf{v}) + T(\mathbf{u}+\mathbf{v})$$

(definition of addition of functions)

$$= S(\mathbf{u}) + S(\mathbf{v}) + T(\mathbf{u}) + T(\mathbf{v})$$

(each of S and T is a linear transformation)

$$= S(\mathbf{u}) + T(\mathbf{u}) + S(\mathbf{v}) + T(\mathbf{v})$$

(commutative law in W)

$= (S+T)(\mathbf{u}) + (S+T)(\mathbf{v})$ Similarly, if α and c are any scalars,

then $(\alpha S)(c\mathbf{u}) = \alpha \times S(c\mathbf{u}) = \alpha \times cS(\mathbf{u})$