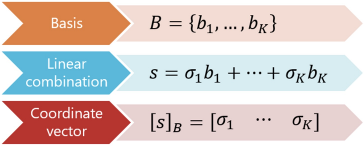
MATH 200C: Linear Algebra



Class 22: Monday, April 6, 2026

- Notes on Assignment 20
- Coordinate Systems

Exam 2: Wednesday, April 8

7 PM – ?

No Calculators, Computers, Phones, Smart Watches, ...

BUT One Sheet of Notes

Last Name	Room
A – K	Warner 105
L – Z	Warner 104

Department of Mathematics and Statistics

Pre-registration Dessert Social

Wednesday, 4/15 | 3:30-4:30pm | Warner 105

Interested in taking some Math or Stat courses in **Fall 2026?** Currently taking a Math or Stats class? Need a study break?

Join the Math & Stats faculty over dessert to:

- Learn about Fall 2026 course offerings
- Get information about:
  - Major in Mathematics and/or the Applied Math Track
  - Major in Statistics
  - Minor in Mathematics
- Ask questions and receive advice about how Math and Stats fits into your Middlebury experience
- Be in community and hear from other students about Math and Stat courses

Anyone who is currently taking or wants to take a Math or Stats course is welcome! Even if you're graduating in May, we hope to see you at the dessert social!

Old Definition: [Section 1.8] A **linear transformation T from $\mathbb{R}^n$ into $\mathbb{R}^m$** is a rule that assigns to each vector $\mathbf{x}$ in $\mathbb{R}^n$ a unique vector $T(\mathbf{x})$ in $\mathbb{R}^m$ such that

$$T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}) \text{ for all } \mathbf{u}, \mathbf{v} \text{ in } \mathbb{R}^n$$

and

$$T(c\mathbf{u}) = cT(\mathbf{u}) \text{ for all } \mathbf{u} \text{ in } V \text{ and all scalars } c.$$

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Definition: A **linear transformation T from a vector space V into a vector space W** is a rule that assigns to each vector $\mathbf{x}$ in V a unique vector $T(\mathbf{x})$ in W such that

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Vector Spaces

Linearly Independent Sets and Bases

Definition: An indexed set of vectors $\{\mathbf{v_1}, \ldots, \mathbf{v_p}\}$ in a vector space V is **linearly independent** if the equation

$$\textbf{(1)} \qquad c_1 v_1 + c_2 v_2 + \cdots + c_p v_p = 0$$

has only the trivial solution, $c_1 = 0, \ldots, c_p = 0$

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The set $\{\mathbf{v_1}, \ldots, \mathbf{v_p}\}$ is said to be **linearly dependent** if **(1)** has a nontrivial solution, that is, if there are some weights, $c_1, \ldots, c_p$, not all zero, such that (1) holds.

In such a case, **(1)** is called a **linear dependence relation** among $\mathbf{v_1}, \ldots, \mathbf{v_p}$

Theorem 4: An indexed set $\{\mathbf{v_1}, \ldots, \mathbf{v_p}\}$ of two or more vectors, with $\mathbf{v_1} \neq 0$, is linearly dependent if and only if some $\mathbf{v_j}$ (with $j > 1$) is a linear combination of the preceding vectors, $\mathbf{v_1}, \ldots, \mathbf{v_{j-1}}$

Definition Let H be a subspace of a vector space V. A set of vectors $\mathcal{B}$ in V is a **basis** for H if

(i) $\mathcal{B}$ is a linearly independent set, and

(ii) the subspace spanned by $\mathcal{B}$ coincides with H; that is, $H = \text{Span } \mathcal{B}$.

Theorem 5: The Spanning Set Theorem: Let $S = \{\mathbf{v_1} \ldots, \mathbf{v_p}\}$ be a set in a vector space V, and let $H = \text{Span}\ \{\mathbf{v_1}, \ldots, \mathbf{v_p}\}$. If one of the vectors in S — say, v_k —is a linear combination of the remaining vectors in S, then the set formed from S by removing $\mathbf{v_k}$ still spans H.

If $H \neq 0$, some subset of S is a basis for H.

Theorem 6: The pivot columns of a matrix A form a basis for $Col\ A$.

Theorem 7: If two matrices A and B are row equivalent, then their row spaces are the same.

If B is in echelon form, the nonzero rows of B form a basis for the row space of A as well as for that of B.

Vector Spaces

Coordinate Systems

Theorem 8: The Unique Representation Theorem: Let $\mathcal{B} = \{\mathbf{b_1}, \ldots, \mathbf{b_n}\}$ be a basis for a vector space V. Then for each $\mathbf{x}$ in V, there exists a unique set of scalars $c_1, \ldots, c_n$ such that

$$\mathbf{x} = c_1\mathbf{b_1} + c_2\mathbf{b_2} + \cdots + c_n\mathbf{b_n}$$

Definition: Suppose $\mathcal{B} = \{\mathbf{b_1}, \ldots, \mathbf{b_n}\}$ be a basis for a vector space V and $\mathbf{x}$ is in V.

The coordinates of x relative to the basis $\mathcal{B}$ (or the $\mathcal{B}$-coordinates of $\mathbf{x}$) are the weights $c_1 \ldots, , c_n$ such that

$$\mathbf{x} = c_1\mathbf{b_1} + c_2\mathbf{b_2} + \cdots + c_n\mathbf{b_n}$$

If $c_1 \ldots, , c_n$ are the $\mathcal{B}$-coordinates of $\mathbf{x}$ then the vector in $\mathbb{R}^{\ltimes}$

$$[\mathbf{x}]_{\mathcal{B}} = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$$

is the **coordinate vector of x (relative to $\mathcal{B}$)**, or the **$\mathcal{B}$-coordinate vector of x** .

The mapping $\mathbf{x} \mapsto \mathbf{x}\mathcal{B}$ is the coordinate mapping (determined by $\mathcal{B}$)

Coordinates in $\mathbb{R}^n$

Let $\mathcal{B} = \{\mathbf{b_1}, \ldots, \mathbf{b_n}\}$ be a basis for $\mathbb{R}^n$.
Let $P_{\mathcal{B}} = [\mathbf{b_1}\ \mathbf{b_2}\ \ldots\ \mathbf{b_n}]$.
Then the vector equation $\mathbf{x} = c_1\mathbf{b_1} + \cdots + c_n\mathbf{b_n}$ is equivalent to

$$\mathbf{x} = P_{\mathcal{B}}[\mathbf{x}]_{\mathcal{B}}$$

We call $P_{\mathcal{B}}$ the change-of-coordinates matrix from $\mathcal{B}$ to the standard basis in $\mathbb{R}^n$.
Left-multiplication by $P_{\mathcal{B}}$ transforms the coordinate vector $[\mathbf{x}]_{\mathcal{B}}$ into $\mathbf{x}$.

Theorem 9: Let $\mathcal{B} = \{\mathbf{b_1}, \ldots, \mathbf{b_n}\}$ be a basis for a vector space V. Then the coordinate mapping $\mathbf{x} \mapsto [\mathbf{x}]_B$ is a one-to-one linear transformation from V onto $\mathbb{R}^n$

The coordinate mapping in Theorem 9 is an important example of an isomorphism V onto $\mathbb{R}^n$.

In general, a one-to-one linear transformation from a vector space V onto a vector space W is called an **isomorphism** from V onto W (*iso* from the Greek for "the same," and *morph* from the Greek for "form" or "structure"). The notation and terminology for V and W may differ, but the two spaces are indistinguishable as vector spaces. Every vector space calculation in V is accurately reproduced in W, and vice versa. In particular, any real vector space with a basis of n vectors is indistinguishable from $\mathbb{R}^n$. (From Lay, *Linear Algebra*).

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If f and g are two ordinary functions, how do you add them?

$$(f+g)(x) = f(x) + g(x)$$

How do you multiply a function by a scalar c?

$$(cf)(x) = c \times f(x)$$

Let V and W be a fixed pair of vector. spaces.
Let $\mathbb{T}$ be the set of all linear transformations from V to W
Suppose S and T are members of $\mathbb{T}$ Then
$(S+T)(\mathbf{u}+\mathbf{v}) = S(\mathbf{u}+\mathbf{v}) + T(\mathbf{u}+\mathbf{v})$
(definition of addition of functions)

$= S(\mathbf{u}) + S(\mathbf{v}) + T(\mathbf{u}) + T(\mathbf{v})$
(each of S and T is a linear transformation)

$= S(\mathbf{u}) + T(\mathbf{u}) + S(\mathbf{v}) + T(\mathbf{v})$
(commutative law in W)

$= (S+T)(\mathbf{u}) + (S+T)(\mathbf{v})$ Similarly, if α and c are any scalars,

then $(\alpha S)(c\mathbf{u}) = \alpha \times S(c\mathbf{u}) = \alpha \times cS(\mathbf{u})$

The set of all linear transformations from V to W is a Vector Space