MATH 200C: Linear Algebra

Dr. Dimensionpants defends his hometown of Ganderville from the constant threat of inter-dimensional villains along with Philip, his talking unicorn pal.

Class 23: Wednesday, April 8, 2026

- Dimension of a Vector Space

UPDATES & ANNOUNCEMENTS

Exam 2: Tonight

7 PM – ?

No Calculators, Computers, Phones, Smart Watches, ...

BUT One Sheet of Notes

Last Name	Room
A – K	Warner 105
L – Z	Warner 104

Department of Mathematics and Statistics

Pre-registration Dessert Social

Wednesday, 4/15 | 3:30-4:30pm | Warner 105

Interested in taking some Math or Stat courses in **Fall 2026?** Currently taking a Math or Stats class? Need a study break?

Join the Math & Stats faculty over dessert to:

- Learn about Fall 2026 course offerings
- Get information about:
  - Major in Mathematics and/or the Applied Math Track
  - Major in Statistics
  - Minor in Mathematics
- Ask questions and receive advice about how Math and Stats fits into your Middlebury experience
- Be in community and hear from other students about Math and Stat courses

Anyone who is currently taking or wants to take a Math or Stats course is welcome! Even if you're graduating in May, we hope to see you at the dessert social!

Vector Spaces

Linearly Independent Sets and Bases

Definition: An indexed set of vectors $\{\mathbf{v_1}, \ldots, \mathbf{v_p}\}$ in a vector space V is **linearly independent** if the equation

$$\textbf{(1)} \qquad c_1 v_1 + c_2 v_2 + \cdots + c_p v_p = 0$$

has only the trivial solution, $c_1 = 0, \ldots, c_p = 0$

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The set $\{\mathbf{v_1}, \ldots, \mathbf{v_p}\}$ is said to be **linearly dependent** if **(1)** has a nontrivial solution, that is, if there are some weights, $c_1, \ldots, c_p$, not all zero, such that (1) holds.

In such a case, **(1)** is called a **linear dependence relation** among $\mathbf{v_1}, \ldots, \mathbf{v_p}$

Theorem 4: An indexed set $\{\mathbf{v_1}, \ldots, \mathbf{v_p}\}$ of two or more vectors, with $\mathbf{v_1} \neq 0$, is linearly dependent if and only if some $\mathbf{v_j}$ (with $j > 1$) is a linear combination of the preceding vectors, $\mathbf{v_1}, \ldots, \mathbf{v_{j-1}}$

Definition Let H be a subspace of a vector space V. A set of vectors $\mathcal{B}$ in V is a **basis** for H if

(i) $\mathcal{B}$ is a linearly independent set, and

(ii) the subspace spanned by $\mathcal{B}$ coincides with H; that is, $H = \text{Span } \mathcal{B}$.

Theorem 5: The Spanning Set Theorem: Let $S = \{\mathbf{v_1} \ldots, \mathbf{v_p}\}$ be a set in a vector space V, and let $H = \text{Span}\ \{\mathbf{v_1}, \ldots, \mathbf{v_p}\}$. If one of the vectors in S — say, v_k —is a linear combination of the remaining vectors in S, then the set formed from S by removing $\mathbf{v_k}$ still spans H.

If $H \neq 0$, some subset of S is a basis for H.

Theorem 6: The pivot columns of a matrix A form a basis for $Col\ A$.

Theorem 7: If two matrices A and B are row equivalent, then their row spaces are the same.

If B is in echelon form, the nonzero rows of B form a basis for the row space of A as well as for that of B.

Vector Spaces

Coordinate Systems

Theorem 8: The Unique Representation Theorem: Let $\mathcal{B} = \{\mathbf{b_1}, \ldots, \mathbf{b_n}\}$ be a basis for a vector space V. Then for each $\mathbf{x}$ in V, there exists a unique set of scalars $c_1, \ldots, c_n$ such that

$$\mathbf{x} = c_1\mathbf{b_1} + c_2\mathbf{b_2} + \cdots + c_n\mathbf{b_n}$$

Definition: Suppose $\mathcal{B} = \{\mathbf{b_1}, \dots, \mathbf{b_n}\}$ be a basis for a vector space V and $\mathbf{x}$ is in V.

The coordinates of x relative to the basis $\mathcal{B}$ (or the $\mathcal{B}$-coordinates of $\mathbf{x}$) are the weights $c_1 \dots, , c_n$ such that

$$\mathbf{x} = c_1\mathbf{b_1} + c_2\mathbf{b_2} + \cdots + c_n\mathbf{b_n}$$

If $c_1 \dots, , c_n$ are the $\mathcal{B}$-coordinates of $\mathbf{x}$ then the vector in $\mathbb{R}^\ltimes$

$$[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

is the **coordinate vector of x (relative to $\mathcal{B}$)**, or the **$\mathcal{B}$-coordinate vector of x** .

The mapping $\mathbf{x} \mapsto \mathbf{x}\mathcal{B}$ is the coordinate mapping (determined by $\mathcal{B}$)

Coordinates in $\mathbb{R}^n$

Let $\mathcal{B} = \{\mathbf{b_1}, \ldots, \mathbf{b_n}\}$ be a basis for $\mathbb{R}^n$.
Let $P_{\mathcal{B}} = [\mathbf{b_1}\ \mathbf{b_2}\ \ldots\ \mathbf{b_n}]$.
Then the vector equation $\mathbf{x} = c_1\mathbf{b_1} + \cdots + c_n\mathbf{b_n}$ is equivalent to
$$\mathbf{x} = P_{\mathcal{B}}[\mathbf{x}]_{\mathcal{B}}$$

We call $P_{\mathcal{B}}$ the change-of-coordinates matrix from $\mathcal{B}$ to the standard basis in $\mathbb{R}^n$.
Left-multiplication by $P_{\mathcal{B}}$ transforms the coordinate vector $[\mathbf{x}]_{\mathcal{B}}$ into $\mathbf{x}$.

Theorem 9: Let $\mathcal{B} = \{\mathbf{b_1}, \ldots, \mathbf{b_n}\}$ be a basis for a vector space V. Then the coordinate mapping $\mathbf{x} \mapsto [\mathbf{x}]_B$ is a one-to-one linear transformation from V onto $\mathbb{R}^n$

The coordinate mapping in Theorem 9 is an important example of an isomorphism V onto $\mathbb{R}^n$.

A one-to-one linear transformation from a vector space V onto a vector space W is called an **isomorphism** from V onto W. The notation and terminology for V and W may differ, but the two spaces are indistinguishable as vector spaces. Every vector space calculation in V is accurately reproduced in W, and vice versa. In particular, any real vector space with a basis of n vectors is indistinguishable from $\mathbb{R}^n$. (From Lay, *Linear Algebra*).

Problem From Last Class

Given the coordinates of a vector with respect to some basis $\mathcal{B}$, find the coordinates of that same vector with respect to a different basis $\mathcal{C}$.

Example: $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \end{bmatrix} \right\}$

and the $\mathcal{B}$ coordinates of a vector $\mathbf{z}$ are $\begin{bmatrix} c \\ d \end{bmatrix}$; that is,

$$[\mathbf{z}]_{\mathcal{B}} = \begin{bmatrix} c \\ d \end{bmatrix}$$

What is $[\mathbf{z}]_{\mathcal{C}}$ if $\mathcal{C} = \left\{ \begin{bmatrix} 2 \\ 11 \end{bmatrix}, \begin{bmatrix} -5 \\ 7 \end{bmatrix} ? \right\}$

Let M be the matrix associated with $\mathcal{B}$.
Let N be the matrix associated with $\mathcal{C}$.

$$M = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \text{ and } N = \begin{bmatrix} 2 & 5 \\ 11 & 7 \end{bmatrix}$$

$$\text{Then } \mathbf{z} = M \begin{bmatrix} c \\ d \end{bmatrix} \text{ and } \mathbf{z} = N \begin{bmatrix} r \\ s \end{bmatrix}$$

$$\text{Thus } N \begin{bmatrix} r \\ s \end{bmatrix} = M \begin{bmatrix} c \\ d \end{bmatrix}$$

How can we find $\begin{bmatrix} r \\ s \end{bmatrix}$?

Dimension of a Vector Space

Theorem 10: If a vector space V has a basis $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$, then any set in V containing more than n vectors must be linearly dependent.

Theorem 10 also applies to infinite sets in V. An infinite set is said to be linearly dependent if some finite subset is linearly dependent; otherwise, the set is linearly independent. If S is an infinite set in V, take any subset $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ of S, with $p > n$. The proof above shows that this subset is linearly dependent and hence so is S."

Theorem 11: If a vector space V has a basis of n vectors, then every basis of V must consist of exactly n vectors."

Definition: If a vector space V is spanned by a finite set, then V is said to be **finite-dimensional**, and the **dimension** of V, written as **dim** V, is the number of vectors in a basis for V. The dimension of the zero vector space $\{0\}$ is defined to be zero. If V is not spanned by a finite set, then V is said to be **infinite-dimensional**.

Subspaces of a Finite-Dimensional Space

Theorem 12: Let H be a subspace of a finite-dimensional vector space V. Any linearly independent set in H can be expanded, if necessary, to a basis for H. Also, H is finite-dimensional and dim $H \leq$ dim V

Theorem 13 The Basis Theorem Let V be a p-dimensional vector space, $p \geq 1$. Any linearly independent set of exactly p elements in V is automatically a basis for V. Any set of exactly p elements that spans V is automatically a basis for V.

The Dimensions of Nul A, Col A, and Row A

Definition: The **rank** of an $m \times n$ matrix A is the dimension of the column space and the nullity of A is the dimension of the null space."

Theorem 14 The Rank Theorem The dimensions of the column space and the null space of an

$m \times n$ matrix A satisfy the equation rank A + nullity A=number of columns in A

Rank and the Invertible Matrix Theorem

Theorem The Invertible Matrix Theorem (continued) Let A be an $n \times n$ matrix. Then the following statements are each equivalent to the statement that A is an invertible matrix.

The columns of A form a basis of $\mathbb{R}^n$

Col $A = \mathbb{R}^n$

rank A=n

nullity A=0

Nul $A = \{0\}$

If f and g are two ordinary functions, how do you add them?

$$(f+g)(x) = f(x) + g(x)$$

How do you multiply a function by a scalar c?

$$(cf)(x) = c \times f(x)$$