MATH 200C: Linear Algebra

Class 24: Friday, April 10, 2026

- Change of Basis
- Notes on Assignment 21
- Some Notes on Exam 2
- Project 2
- Team Assignments for Project 2

Exam 3	Wednesday, May 6	20%
Project	Monday, May 11	5%
Final Exam	Thursday, May 14	30 %

Department of Mathematics and Statistics

Pre-registration Dessert Social

Wednesday, 4/15 | 3:30-4:30pm | Warner 105

Interested in taking some Math or Stat courses in **Fall 2026?** Currently taking a Math or Stats class? Need a study break?

Join the Math & Stats faculty over dessert to:

- Learn about Fall 2026 course offerings
- Get information about:
  - Major in Mathematics and/or the Applied Math Track
  - Major in Statistics
  - Minor in Mathematics
- Ask questions and receive advice about how Math and Stats fits into your Middlebury experience
- Be in community and hear from other students about Math and Stat courses

Anyone who is currently taking or wants to take a Math or Stats course is welcome! Even if you're graduating in May, we hope to see you at the dessert social!

Vector Spaces
Change of Bases

Vector Spaces

Coordinate Systems

Theorem 8: The Unique Representation Theorem: Let $\mathcal{B} = \{\mathbf{b_1}, \ldots, \mathbf{b_n}\}$ be a basis for a vector space V. Then for each $\mathbf{x}$ in V, there exists a unique set of scalars $c_1, \ldots, c_n$ such that

$$\mathbf{x} = c_1\mathbf{b_1} + c_2\mathbf{b_2} + \cdots + c_n\mathbf{b_n}$$

<u>Definition</u>: Suppose $\mathcal{B} = \{\mathbf{b_1}, \ldots, \mathbf{b_n}\}$ be a basis for a vector space V and $\mathbf{x}$ is in V.

The coordinates of x relative to the basis $\mathcal{B}$ (or the $\mathcal{B}$-coordinates of $\mathbf{x}$) are the weights $c_1 \ldots, , c_n$ such that

$$\mathbf{x} = c_1\mathbf{b_1} + c_2\mathbf{b_2} + \cdots + c_n\mathbf{b_n}$$

If $c_1 \ldots, , c_n$ are the $\mathcal{B}$-coordinates of $\mathbf{x}$ then the vector in $\mathbb{R}^{\ltimes}$

$$[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

is the **coordinate vector of x (relative to $\mathcal{B}$)**, or the **$\mathcal{B}$-coordinate vector of x** .

The mapping $\mathbf{x} \mapsto \mathbf{x}\mathcal{B}$ is the coordinate mapping (determined by $\mathcal{B}$)

Coordinates in $\mathbb{R}^n$

Let $\mathcal{B} = \{\mathbf{b_1}, \ldots, \mathbf{b_n}\}$ be a basis for $\mathbb{R}^n$.
Let $P_{\mathcal{B}} = [\mathbf{b_1}\ \mathbf{b_2}\ \ldots\ \mathbf{b_n}]$.
Then the vector equation $\mathbf{x} = c_1\mathbf{b_1} + \cdots + c_n\mathbf{b_n}$ is equivalent to
$\mathbf{x} = P_{\mathcal{B}}[\mathbf{x}]_{\mathcal{B}}$

We call $P_{\mathcal{B}}$ the change-of-coordinates matrix from $\mathcal{B}$ to the standard basis in $\mathbb{R}^n$.
Left-multiplication by $P_{\mathcal{B}}$ transforms the coordinate vector $[\mathbf{x}]_{\mathcal{B}}$ into $\mathbf{x}$.

Theorem 9: Let $\mathcal{B} = \{\mathbf{b_1}, \ldots, \mathbf{b_n}\}$ be a basis for a vector space V. Then the coordinate mapping $\mathbf{x} \mapsto [\mathbf{x}]_B$ is a one-to-one linear transformation from V onto $\mathbb{R}^n$

The coordinate mapping in Theorem 9 is an important example of an isomorphism V onto $\mathbb{R}^n$.

A one-to-one linear transformation from a vector space V onto a vector space W is called an **isomorphism** from V onto W . The notation and terminology for V and W may differ, but the two spaces are indistinguishable as vector spaces. Every vector space calculation in V is accurately reproduced in W, and vice versa. In particular, any real vector space with a basis of n vectors is indistinguishable from $\mathbb{R}^n$. (From Lay, *Linear Algebra*).

Vector Spaces

Change of Basis

Theorem 15: Let $\mathcal{B} = \{ \mathbf{b_1}, \dots, \mathbf{b_n} \}$ and $\mathcal{C} = \{ \mathbf{c_1}, \dots, \mathbf{c_n} \}$ be bases for the same vector space V.

There exists a unique $n \times n$ matrix $\underset{\mathcal{C} \leftarrow \mathcal{B}}{P}$ such that

$$[\mathbf{x}]_{\mathcal{C}} = \underset{\mathcal{C} \leftarrow \mathcal{B}}{P} [\mathbf{x}]_{\mathcal{B}}$$

The columns of $\underset{\mathcal{C} \leftarrow \mathcal{B}}{P}$ are the $\mathcal{C}$ -coordinates vectors of the vectors in the basis $\mathcal{B}$; that is,

$$\underset{\mathcal{C} \leftarrow \mathcal{B}}{P} = [\ [\boldsymbol{b_1}]_{\mathcal{C}} \quad [\boldsymbol{b_2}]_{\mathcal{C}} \quad \dots . [\boldsymbol{b_n}]_{\mathcal{C}}\ ]$$

The matrix $\underset{\mathcal{C} \leftarrow \mathcal{B}}{P}$ is called the **change-of-coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$.**

Dimension of a Vector Space

Theorem 10: If a vector space V has a basis $\mathcal{B} = \{ \mathbf{b}_1, \ldots, \mathbf{b}_n \}$, then any set in V containing more than n vectors must be linearly dependent.

Theorem 10 also applies to infinite sets in V. An infinite set is said to be linearly dependent if some finite subset is linearly dependent; otherwise, the set is linearly independent. If S is an infinite set in V, take any subset $\{ \mathbf{u}_1, \ldots, \mathbf{u}_p \}$ of S, with $p > n$. The proof above shows that this subset is linearly dependent and hence so is S."

Theorem 11: If a vector space V has a basis of n vectors, then every basis of V must consist of exactly n vectors."

Definition: If a vector space V is spanned by a finite set, then V is said to be **finite-dimensional**, and the **dimension** of V, written as **dim V**, is the number of vectors in a basis for V. The dimension of the zero vector space $\{0\}$ is defined to be zero. If V is not spanned by a finite set, then V is said to be **infinite-dimensional**.

Subspaces of a Finite-Dimensional Space

Theorem 12: Let H be a subspace of a finite-dimensional vector space V. Any linearly independent set in H can be expanded, if necessary, to a basis for H. Also, H is finite-dimensional and dim $H \leq$ dim V

Theorem 13 The Basis Theorem Let V be a p-dimensional vector space, $p \geq 1$. Any linearly independent set of exactly p elements in V is automatically a basis for V. Any set of exactly p elements that spans V is automatically a basis for V.

The Dimensions of Nul A, Col A, and Row A

Definition: The **rank** of an $m \times n$ matrix A is the dimension of the column space and the nullity of A is the dimension of the null space."

Theorem 14 The Rank Theorem The dimensions of the column space and the null space of an

$m \times n$ matrix A satisfy the equation rank A + nullity A=number of columns in A

Rank and the Invertible Matrix Theorem

Theorem The Invertible Matrix Theorem (continued) Let A be an $n \times n$ matrix. Then the following statements are each equivalent to the statement that A is an invertible matrix.

The columns of A form a basis of $\mathbb{R}^n$

Col $A = \mathbb{R}^n$

rank A=n

nullity A=0

Nul $A = \{0\}$

If f and g are two ordinary functions, how do you add them?

$$(f+g)(x) = f(x) + g(x)$$

How do you multiply a function by a scalar c?

$$(cf)(x) = c \times f(x)$$