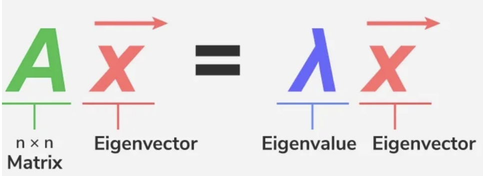
MATH 200C: Linear Algebra



Class 25: Monday, April 13, 2026

- Eigenvalues and Eigenvectors
- Notes on Assignment 22

Exam 3	Wednesday, May 6	20%
Project	Monday, May 11	5%
Final Exam	Thursday, May 14	30 %

Department of Mathematics and Statistics

Pre-registration Dessert Social

Wednesday, 4/15 | 3:30-4:30pm | Warner 105

Interested in taking some Math or Stat courses in **Fall 2026?** Currently taking a Math or Stats class? Need a study break?

Join the Math & Stats faculty over dessert to:

- Learn about Fall 2026 course offerings
- Get information about:
  - Major in Mathematics and/or the Applied Math Track
  - Major in Statistics
  - Minor in Mathematics
- Ask questions and receive advice about how Math and Stats fits into your Middlebury experience
- Be in community and hear from other students about Math and Stat courses

Anyone who is currently taking or wants to take a Math or Stats course is welcome! Even if you're graduating in May, we hope to see you at the dessert social!

Begin With Question 6 on Exam 2

$$M = \begin{bmatrix} -17 & -30 \\ 10 & 18 \end{bmatrix}, \ \{\mathbf{u}, \mathbf{v}\} \text{ is basis for } \mathbb{R}^2$$

$$\mathbf{u} = \begin{bmatrix} -3 \\ 2 \end{bmatrix} \text{ with } M\mathbf{u} = 3\mathbf{u}, \ \mathbf{v} = \begin{bmatrix} -2 \\ 1 \end{bmatrix} \text{ with } M\mathbf{v} = -2\mathbf{v}$$

Let $\mathbf{x}$ be any vector in $\mathbb{R}^2$. Then $\mathbf{x} = c_1\mathbf{u} + c_2\mathbf{v}$

Now $M\mathbf{x} = M(c_1\mathbf{u} + c_2\mathbf{v}) = M(c_1\mathbf{u}) + M(c_2\mathbf{v}) =$

$c_1 M(\mathbf{u}) + c_2 M(\mathbf{v}) = c_1(3\mathbf{u}) + c_2(-2\mathbf{v})$

$M^2\mathbf{x} = M(M\mathbf{x}) = M(c_1(3\mathbf{u}) + c_2(-2\mathbf{v})) = c_1(3^2)\mathbf{u} + c_2(-2)^2\mathbf{v}$

$M^k\mathbf{x} = c_1(3^k)\mathbf{u} + c_2(-2)^k\mathbf{v}$

Eigenvalues and Eigenvectors

Definition: An **eigenvector** of an $n \times n$ matrix A is a nonzero vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$ for some scalar λ. A scalar λ is called an **eigenvalue** of A if there is a nontrivial solution $\mathbf{x}$ of $A\mathbf{x} = \lambda\mathbf{x}$; such an x is called an eigenvector corresponding to λ.

$$\text{Example: } A = \begin{bmatrix} -54 & 2 & 50 \\ -48 & 3 & 45 \\ -54 & 2 & 50 \end{bmatrix}$$

$$\mathbf{x} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} ; Q\mathbf{x} = Q \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ 2 \end{bmatrix} = 2\mathbf{x}$$

Thus 2 is an eigenvalue of A with associated eigenvector $\mathbf{x} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$

Example: $A = \begin{bmatrix} -54 & 2 & 50 \\ -48 & 3 & 45 \\ -54 & 2 & 50 \end{bmatrix}$

$$\mathbf{x} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}; Q\mathbf{x} = Q \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -6 \\ -3 \\ -6 \end{bmatrix} = -3\mathbf{x}$$

Thus -3 is an eigenvalue of A with associated eigenvector $\mathbf{x} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$

Example: $A = \begin{bmatrix} -54 & 2 & 50 \\ -48 & 3 & 45 \\ -54 & 2 & 50 \end{bmatrix}$

$$\mathbf{x} = \begin{bmatrix} 10 \\ -5 \\ 11 \end{bmatrix}; Q\mathbf{x} = Q \begin{bmatrix} 10 \\ -5 \\ 11 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0\mathbf{x}$$

Thus 0 is an eigenvalue of A with associated eigenvector

$$\mathbf{x} = \begin{bmatrix} 10 \\ -5 \\ 11 \end{bmatrix}$$

Suppose λ is an eigenvalue of A with an associated eigenvector $\mathbf{x}$. Suppose c is a constant. Then

$$A(c\mathbf{x}) = cA\mathbf{x} = c\lambda\mathbf{x} = \lambda c\mathbf{x}$$

Thus any nonzero scalar multiple of an eigenvector is an eigenvector with the same eigenvalue.

Example Show that $\lambda = -2$ is an eigenvalue of $A = \begin{bmatrix} 52 & 36 \\ -72 & -50 \end{bmatrix}$ and find an associated eigenvector.

Solution $A - \lambda I = A + 2I = \begin{bmatrix} 54 & 36 \\ -72 & -48 \end{bmatrix}$

We want to find a nonzero solution to $\begin{bmatrix} 54 & 36 \\ -72 & -48 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Row reduce the coefficient matrix:

$$\begin{bmatrix} 54 & 36 \\ -72 & -48 \end{bmatrix} \to \begin{bmatrix} 1 & \frac{2}{3} \\ -72 & -48 \end{bmatrix} \to \begin{bmatrix} 1 & \frac{2}{3} \\ 0 & 0 \end{bmatrix}$$

Thus $x_1 + \frac{2}{3}x_2 = 0$ so $x_1 = -\frac{2}{3}x_2$. One such vector is $\mathbf{x} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$

Given that λ is an eigenvalue for a matrix A, we can find an eigenvector by solving the homogenous system of linear equations $(A - \lambda I)\mathbf{x} = \mathbf{0}$ for a nonzero $\mathbf{x}$.

Warning: Although we can use row reduction to find eigenvectors, it cannot be used to find eigenvalues. An echelon form of a matrix A usually does not display the eigenvalues of A.

How do find the eigenvalues of A?

Given that λ is an eigenvalue for a matrix A, we can find an eigenvector by solving the homogenous system of linear equations $(A - \lambda I)\mathbf{x} = \mathbf{0}$ for a nonzero $\mathbf{x}$.

Warning: Although we can use row reduction to find eigenvectors, it cannot be used to find eigenvalues. An echelon form of a matrix A usually does not display the eigenvalues of A.

How do find the eigenvalues of A?
If a nonzero solution to the equation $(A - \lambda I)\mathbf{x} = \mathbf{0}$ exists, then $A - \lambda I$ must be an non-invertible matrix.

Given that λ is an eigenvalue for a matrix A, we can find an eigenvector by solving the homogenous system of linear equations $(A - \lambda I)\mathbf{x} = \mathbf{0}$ for a nonzero $\mathbf{x}$.

Warning: Although we can use row reduction to find eigenvectors, it cannot be used to find eigenvalues. An echelon form of a matrix A usually does not display the eigenvalues of A.

How do find the eigenvalues of A?
If a nonzero solution to the equation $(A - \lambda I)\mathbf{x} = \mathbf{0}$ exists, then $A - \lambda I$ must be an non-invertible matrix.
So $det(A - \lambda I) = 0$

$$\underline{\text{Our Example}}: \ A = \begin{bmatrix} 52 & 36 \\ -72 & -50 \end{bmatrix}$$

$$\text{Now } A - \lambda I = \begin{bmatrix} 52-\lambda & 36 \\ -72 & -50-\lambda \end{bmatrix}$$

which has determinant $(52-\lambda)(-50-\lambda) - (36)(-72) =$ $-2600 - 52\lambda + 50\lambda + \lambda^2 + 2592 = \lambda^2 - 2\lambda - 8 = (\lambda-4)(\lambda+2)$

Theorem 1: The eigenvalues of a triangular matrix are the entries on its main diagonal.

Theorem 2: If $\mathbf{v_1}, \ldots, \mathbf{v_r}$ are eigenvectors that correspond to distinct eigenvalues $\lambda_1, \ldots, \lambda_r$ of an $n \times n$ matrix A, then the set $\{\mathbf{v_1}, \ldots, \mathbf{v_r}\}$ is linearly independent.

A scalar λ is an eigenvalue of a square matrix A if and only if λ satisfies the characteristic equation $\det(A - \lambda I) = 0$.

Let V and W be a fixed pair of vector. spaces.
Let $\mathbb{T}$ be the set of all linear transformations from V to W
Suppose S and T are members of $\mathbb{T}$ Then
$(S + T)(\mathbf{u} + \mathbf{v}) = S(\mathbf{u} + \mathbf{v}) + T(\mathbf{u} + \mathbf{v})$
(definition of addition of functions)

$= S(\mathbf{u}) + S(\mathbf{v}) + T(\mathbf{u}) + T(\mathbf{v})$
(each of S and T is a linear transformation)

$= S(\mathbf{u}) + T(\mathbf{u}) + S(\mathbf{v}) + T(\mathbf{v})$
(commutative law in W)

$= (S + T)(\mathbf{u}) + (S + T)(\mathbf{v})$ Similarly, if α and c are any scalars,

then $(\alpha S)(c\mathbf{u}) = \alpha \times S(c\mathbf{u}) = \alpha \times cS(\mathbf{u})$

The set of all linear transformations from V to W is a Vector Space