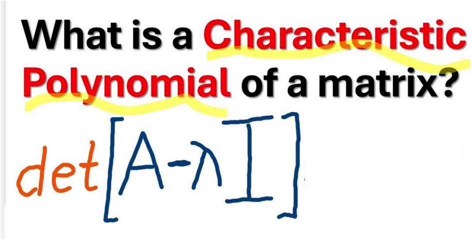
MATH 200C: Linear Algebra



Class 26: Wednesday, April 15, 2026

- Characteristic Equation
- Notes on Assignment 23

TODAY

MATH & STATS
PRE-REGISTRATION DESSERT SOCIAL

Wed, April 15 | 3:30-4:30 PM | Warner 105

Are you interested in registering for Mathematics and Statistics courses in Fall 2026 but don't know which ones to take? Are you curious about the Math and Stats majors and minor options? Come chat with Mathematics and Statistics faculty over dessert to learn about the amazing courses we have on offer. Discover how math and stats might fit into your Middlebury experience, or just to say hi! We'll have plenty of cookies, ice-cream, fruit, and advice to share!

Eigenvalues and Eigenvectors

Definition: An **eigenvector** of an $n \times n$ matrix A is a nonzero vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$ for some scalar λ.

A scalar λ is called an **eigenvalue** of A if there is a nontrivial solution $\mathbf{x}$ of $A\mathbf{x} = \lambda\mathbf{x}$; such an x is called an eigenvector corresponding to λ.

Suppose λ is an eigenvalue of an $n \times n$ matrix A with an associated eigenvector $\mathbf{x}$. Suppose c is a constant. Then

$$A(c\mathbf{x}) = cA\mathbf{x} = c\lambda\mathbf{x} = \lambda c\mathbf{x}$$

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Thus any nonzero scalar multiple of an eigenvector is an eigenvector with the same eigenvalue.

If $\mathbf{x}$ and $\mathbf{y}$ are two eigenvectors associated with the same eigenvalue λ of A, then $A(\mathbf{x}+\mathbf{y}) = A\mathbf{x} + A\mathbf{y} = \lambda\mathbf{x} + \lambda\mathbf{y} = \lambda(\mathbf{x}+\mathbf{y})$ so $\mathbf{x}+\mathbf{y}$ is also an eigenvector associated with the same eigenvalue λ

The set of eigenvectors of A associated with eigenvalue λ is closed under scalar multiplication and vector addition.

Conclusion: S is a subspace of $\mathbb{R}^n$.

The collection of eigenvectors of A associated with eigenvalue λ is closed under scalar multiplication and vector addition.

Let S be the **zero vector** and all the eigenvectors of A associated with eigenvalue λ

Then S is a subspace of $\mathbb{R}^n$.

The dimension of S is called the **geometric multiplicity** of λ.

Given that λ is an eigenvalue for a matrix A, we can find an eigenvector by solving the homogenous system of linear equations $(A - \lambda I)\mathbf{x} = \mathbf{0}$ for a nonzero $\mathbf{x}$.

S is the set of solutions of the system $(A - \lambda I)\mathbf{x} = \mathbf{0}$

S is called the **eigenspace** of λ.

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So $det(A - \lambda I) = 0$

Theorem 2: If $\mathbf{v_1}, \ldots, \mathbf{v_r}$ are eigenvectors that correspond to distinct eigenvalues $\lambda_1, \ldots, \lambda_r$ of an $n \times n$ matrix A, then the set $\{\mathbf{v_1}, \ldots, \mathbf{v_r}\}$ is linearly independent.

A scalar λ is an eigenvalue of a square matrix A if and only if λ satisfies the **characteristic equation** $\det(A - \lambda I) = 0$.

For an $n \times n$ matrix, $\det(A - \lambda I)$ is a polynomial of degree n, called the **characteristic polynomial**

Definition: An eigenvalue r has **algebraic multiplicity** k if k is the largest integer such that

Example 1: $A = \begin{bmatrix} 8 & 0 & 0 & -12 \\ -6 & 2 & 0 & 12 \\ 6 & 1 & 3 & -12 \\ 6 & 0 & 0 & -10 \end{bmatrix}$ has

characteristic polynomial

$$\lambda^4 - 3\lambda^3 - 12\lambda^2 + 52\lambda - 48 = (\lambda + 4)(\lambda - 3)(\lambda - 2)^2$$

Eigenvalue	Algebraic Multiplicity	Geometric Multiplicity
-4	1	1
3	1	1
2	2	2

$\left\{ \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for the eigenspace for $\lambda = 2$; that is, the

solution space of $(A - 2\lambda I)\mathbf{x} = \mathbf{0}$.

Example 2: $A = \begin{bmatrix} 2 & 1 & -1 & 1 \\ 1 & 2 & -1 & 0 \\ 0 & 0 & 1 & 1 \\ -1/2 & 3/2 & -3/2 & 4 \end{bmatrix}$ has characteristic polynomial

$$\lambda^4 - 9\lambda^3 + 29\lambda^2 - 39\lambda + 18 = (\lambda - 1)(\lambda - 2)(\lambda - 3)^2$$

Eigenvalue	Algebraic Multiplicity	Geometric Multiplicity
1	1	1
2	1	1
3	2	1

$\left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}$ is a basis for the eigenspace for $\lambda = 3$; that is, the solution space of $(A - 3\lambda I)\mathbf{x} = \mathbf{0}$.

Similar Matrices

Definition: Two $n \times n$ square matrices A and B are **similar** if there is an $n \times n$ invertible matrix P such that $P^{-1}AP = B$.

Theorem: If $n \times n$ square matrices A and BZ are similar, then they have the same characteristic polynomials and thus the same eigenvalues with the same multiplicities.