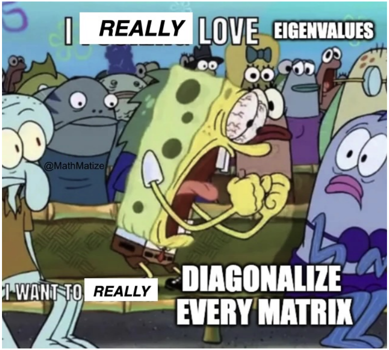
MATH 200C: Linear Algebra



Class 27: Monday, April 20, 2026

- Diagonalization
- Notes on Assignment 24

Eigenvalues and Eigenvectors

Definition: An **eigenvector** of an $n \times n$ matrix A is a nonzero vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$ for some scalar λ.

A scalar λ is called an **eigenvalue** of A if there is a nontrivial solution $\mathbf{x}$ of $A\mathbf{x} = \lambda\mathbf{x}$; such an x is called an eigenvector corresponding to λ.

Theorem 2: If $\mathbf{v_1}, \ldots, \mathbf{v_r}$ are eigenvectors that correspond to distinct eigenvalues $\lambda_1, \ldots, \lambda_r$ of an $n \times n$ matrix A, then the set $\{\mathbf{v_1}, \ldots, \mathbf{v_r}\}$ is linearly independent.

From Last Time:

Suppose $\lambda_1, \lambda_2, \lambda_3$ are distinct eigenvalues of a square matrix A with associated eigenvectors $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$

Thus $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$ are nonzero vectors

$$A\mathbf{v_1} = \lambda_1 \mathbf{v_1}, A\mathbf{v_2} = \lambda_2 \mathbf{v_2}, A\mathbf{v_3} = \lambda_3 \mathbf{v_3}$$

$$\lambda_1 \neq \lambda_2, \lambda_1 \neq \lambda_3, \lambda_2 \neq \lambda_3$$

Suppose $\{\mathbf{v_1}, \mathbf{v_2}\}$ is linearly independent, but $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ is linearly dependent.

Then (*) $\mathbf{v_3} = a_1\mathbf{v_1} + a_2\mathbf{v_2}$ for some constants a_1, a_2

Multiply (*) by λ_3 to obtain

Ian Equation: $\lambda_3\mathbf{v_3} = a_1\lambda_3\mathbf{v_1} + a_2\lambda_3\mathbf{v_2}$

Multiply (*) by A to obtain Lambda Equation

$$A\mathbf{v_3} = A(a_1\mathbf{v_1} + a_2\mathbf{v_2}) = Aa_1\mathbf{v_1} + Aa_2\mathbf{v_2} = a_1\lambda_1\mathbf{v_1} + a_2\lambda_2\mathbf{v_2}$$

$$A\mathbf{v_3} = a_1\lambda_1\mathbf{v_1} + a_2\lambda_2\mathbf{v_2}$$

Left Hand side equals $\lambda_3\mathbf{v_3}$ so **Lambda Equation** is now

$$\lambda_3\mathbf{v_3} = a_1\lambda_1\mathbf{v_1} + a_2\lambda_2\mathbf{v_2}$$

Recall (*) $\mathbf{v_3} = a_1\mathbf{v_1} + a_2\mathbf{v_2}$

Ian Equation: $\lambda_3\mathbf{v_3} = a_1\lambda_3\mathbf{v_1} + a_2\lambda_3\mathbf{v_2}$

Lambda Equation: $\lambda_3\mathbf{v_3} = a_1\lambda_1\mathbf{v_1} + a_2\lambda_2\mathbf{v_2}$

Subtract **Lambda Equation** From **Ian Equation**:

$$\mathbf{0} = a_1(\lambda_3 - \lambda_1)\mathbf{v_1} + a_2(\lambda_3 - \lambda_2)\mathbf{v_2}$$

But $\{\mathbf{v_1}, \mathbf{v_2}\}$ is linearly independent

so $a_1(\lambda_3 - \lambda_1) = 0$ and $a_2(\lambda_3 - \lambda_2) = 0$

Since $(\lambda_3 - \lambda_1) \neq 0$ and $(\lambda_3 - \lambda_2) \neq 0$, we must have

$$a_1 = 0, a_2 = 0$$

which makes (*) $\mathbf{v_3} = 0\mathbf{v_1} + 0\mathbf{v_2} = \mathbf{0}$ which contradicts $\mathbf{v_3} \neq \mathbf{0}$

Proof in General Case

Suppose $\lambda_1, \lambda_2, \ldots, \lambda_p, \lambda_{p+1}$ are distinct eigenvalues of a square matrix A with associated eigenvectors $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_p}, \mathbf{v_{p+1}}$,

Thus $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_p}, \mathbf{v_{p+1}}$, are all nonzero vectors

$A\mathbf{v_i} = \lambda_i \mathbf{v_i}$ for $i = 1, 2, \ldots, p, p+1$,

$\lambda_i \neq \lambda_j$ if $i \neq j$

Suppose $\{\mathbf{v_1}, \ldots \mathbf{v_p}\}$ is linearly independent, but $\{\mathbf{v_1}, \ldots, \mathbf{v_p}, \mathbf{v_{p+1}}\}$ is linearly dependent.

Then (*) $\mathbf{v_{p+1}} = a_1\mathbf{v_1} + a_2\mathbf{v_2} + \cdots + a_p\mathbf{v_p}$ for some constants $a_1, a_2, \ldots, a_p$

Multiply (*) by λ_{p+1} to obtain

Ian Equation: $\lambda_{p+1}\mathbf{v_{p+1}} = a_1\lambda_{p+1}\mathbf{v_1} + a_2\lambda_{p+1}\mathbf{v_2} + \cdots + a_p\lambda_{p+1}\mathbf{v_p}$

Multiply (*) by A to obtain Lambda Equation

$A\mathbf{v_{p+1}} = A(a_1\mathbf{v_1} + a_2\mathbf{v_2} + \cdots + a_p\mathbf{v_p}) =$

$Aa_1\mathbf{v_1} + Aa_2\mathbf{v_2} + \cdots + Aa_p\mathbf{v_p} = a_1\lambda_1\mathbf{v_1} + a_2\lambda_2\mathbf{v_2} + \cdots + a_p\lambda_p\mathbf{v_p}$

$$A\mathbf{v_{p+1}} = a_1\lambda_1\mathbf{v_1} + a_2\lambda_2\mathbf{v_2} + \cdots + a_p\lambda_p\mathbf{v_p}$$

Left Hand side equals $\lambda_{p+1}\mathbf{v_{p+1}}$ so **Lambda Equation** is now

$$\lambda_{p+1}\mathbf{v_3} = a_1\lambda_1\mathbf{v_1} + a_2\lambda_2\mathbf{v_2} + \cdots + a_p\lambda_p\mathbf{v_p}$$

Recall (*) $\mathbf{v_{p+1}} = a_1\mathbf{v_1} + a_2\mathbf{v_2} + \cdots + a_p\mathbf{v_p}$ for some constants $a_1, a_2, \ldots, a_p$

Ian Equation: $\lambda_{p+1}\mathbf{v_{p+1}} = a_1\lambda_{p+1}\mathbf{v_1} + a_2\lambda_{p+1}\mathbf{v_2} + \cdots + a_p\lambda_{p+1}\mathbf{v_p}$

Lambda Equation: $\lambda_{p+1}\mathbf{v_{p+1}} = a_1\lambda_1\mathbf{v_1} + a_2\lambda_2\mathbf{v_2} + \cdots + a_p\lambda_p\mathbf{v_p}$

Subtract **Lambda Equation** From **Ian Equation**:

$$\mathbf{0} = a_1(\lambda_{p+1} - \lambda_1)\mathbf{v_1} + a_2(\lambda_{p+1} - \lambda_2)\mathbf{v_2} + a_2(\lambda_{p+1} - \lambda_p)\mathbf{v_p}$$

But $\{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_p}\}$ is linearly independent

so $a_i(\lambda_{p+1} - \lambda_i) = 0$ for $i = 1, 2, \ldots, p$

Since $(\lambda_{p+1} - \lambda_i) \neq 0$ for $i = 1, 2, \ldots, p$

we must have $a_i = 0$ for $i = 1, 2, \ldots, p$

which makes (*) $\mathbf{v_{p+1}} = 0\mathbf{v_1} + 0\mathbf{v_2} + \cdots + 0\mathbf{v_p} = \mathbf{0}$ which contradicts $\mathbf{v_{p+1}} \neq \mathbf{0}$

Similar Matrices

Definition: Two $n \times n$ square matrices A and B are **similar** if there is an $n \times n$ invertible matrix P such that $P^{-1}AP = B$.

Theorem: If $n \times n$ square matrices A and B are similar, then they have the same characteristic polynomials and thus the same eigenvalues with the same multiplicities.

Diagonalization

The eigenvalue–eigenvector information contained within a matrix A often can be displayed in a useful factorization of the form $A = PDP^{-1}$ where D is a diagonal matrix.

The factorization enables us to compute A^k quickly for large values of k, a fundamental idea in several applications of linear algebra.

Theorem 5: The Diagonalization Theorem An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors.

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In fact, $A = PDP^{-1}$, with D a diagonal matrix, if and only if the columns of P are n linearly independent eigenvectors of A.
In this case, the diagonal entries of D are eigenvalues of A that correspond, respectively, to the eigenvectors in P.

Theorem 6: An $n \times n$ matrix with n distinct eigenvalues is diagonalizable.

Theorem 7 Let A be an $n \times n$ matrix whose distinct eigenvalues are $\lambda_1, \ldots, \lambda_p$

- (a) For $1 \leq k \leq p$, the dimension of the eigenspace for λ_k is less than or equal to the multiplicity of the eigenvalue λ_k.
- (b) The matrix A is diagonalizable if and only if the sum of the dimensions of the eigenspaces equals n, and this happens if and only if
  - (i) the characteristic polynomial factors completely into linear factors and
  - (ii) the dimension of the eigenspace for each λ_k equals the algebraic multiplicity of λ_k.
- (c) If A is diagonalizable and B_k is a basis for the eigenspace corresponding to λ_k for each k, then the total collection of vectors in the sets $B_1, \ldots, B_p$ forms an eigenvector basis for R^n.