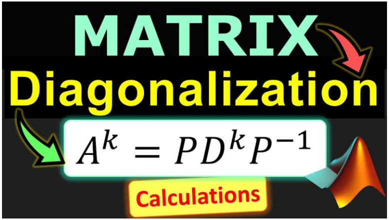
MATH 200C: Linear Algebra



Class 28: Wednesday, April 22, 2026

Handy Handouts

- Eigenvectors and Linear Transformations
- Notes on Assignment 25

Eigenvalues and Eigenvectors

Definition: An **eigenvector** of an $n \times n$ matrix A is a nonzero vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$ for some scalar λ.

A scalar λ is called an **eigenvalue** of A if there is a nontrivial solution $\mathbf{x}$ of $A\mathbf{x} = \lambda\mathbf{x}$; such an x is called an eigenvector corresponding to λ.

Similar Matrices

Definition: Two $n \times n$ square matrices A and B are **similar** if there is an $n \times n$ invertible matrix P such that $P^{-1}AP = B$.

Theorem: If $n \times n$ square matrices A and B are similar, then they have the same characteristic polynomials and thus the same eigenvalues with the same multiplicities.

Diagonalization

The eigenvalue–eigenvector information contained within a matrix A often can be displayed in a useful factorization of the form $A = PDP^{-1}$ where D is a diagonal matrix.

The factorization enables us to compute A^k quickly for large values of k, a fundamental idea in several applications of linear algebra.

Definition: A square matrix A is **diagonalizable** if it is similar to a diagonal matrix D; that is, $A = PDP^{-1}$ for some invertible matrix P.

Theorem 5: The Diagonalization Theorem An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors.

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In fact, $A = PDP^{-1}$, with D a diagonal matrix, if and only if the columns of P are n linearly independent eigenvectors of A.
In this case, the diagonal entries of D are eigenvalues of A that correspond, respectively, to the eigenvectors in P.

Theorem 6: An $n \times n$ matrix with n distinct eigenvalues is diagonalizable.

Theorem 7 Let A be an $n \times n$ matrix whose distinct eigenvalues are $\lambda_1, \ldots, \lambda_p$

- (a) For $1 \leq k \leq p$, the dimension of the eigenspace for λ_k is less than or equal to the multiplicity of the eigenvalue λ_k.
- (b) The matrix A is diagonalizable if and only if the sum of the dimensions of the eigenspaces equals n, and this happens if and only if
  - (i) the characteristic polynomial factors completely into linear factors and
  - (ii) the dimension of the eigenspace for each λ_k equals the algebraic multiplicity of λ_k.
- (c) If A is diagonalizable and B_k is a basis for the eigenspace corresponding to λ_k for each k, then the total collection of vectors in the sets $B_1, \ldots, B_p$ forms an eigenvector basis for R^n.

Eigenvectors and Linear Transformations

Suppose $T : V \to V$ is a linear transformation on a finite dimensional vector space V.

If there is a basis for V consisting of eigenvectors of T, then we will show how to represent the transformation as left multiplication by a diagonal matrix.

Definition Let V be a vector space.
An **eigenvector of a linear transformation** T:$V \to V$ is a nonzero vector $\mathbf{x}$ in V such that $T(\mathbf{x}) = \lambda\mathbf{x}$ for some scalar λ.

A scalar λ is called an **eigenvalue of T** if there is a nontrivial solution $\mathbf{x}$ of $T(\mathbf{x}) = \lambda\,\mathbf{x}$;
We call such an $\mathbf{x}$ an **eigenvector corresponding to** λ .

The Matrix of a Linear Transformation

Let V be an n-dimensional vector space and let T be any linear transformation from V to V.
Choose any basis $\mathcal{B} = \{\mathbf{b_1}, \dots, \mathbf{b_n}\}$ for for V.
Given any $\mathbf{x}$ in V, the coordinate vector $[\mathbf{x}]_{\mathcal{B}}$ is in $\mathbb{R}^n$, as is the coordinate vector of its image, $T(\mathbf{x})]_{\mathcal{B}}$

$$\text{If } \mathbf{x} = r_1\mathbf{b_1} + \cdots + r_n\mathbf{b_n} \text{ then } [\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} r_1 \\ \vdots \\ r_n \end{bmatrix}$$

and

$$T(\mathbf{x}) = T(r_1\mathbf{b_1} + \cdots + r_n\mathbf{b_n}) = r_1T(\mathbf{b_1}) + \cdots + r_nT(\mathbf{b_n})$$

Since the coordinate mapping from V to $\mathbb{R}^n$ is linear (Theorem 8 of Section 4.4), we have

$$T(\mathbf{x}) = T(r_1\mathbf{b_1} + \cdots + r_n\mathbf{b_n}) = r_1T(\mathbf{b_1}) + \cdots + r_nT(\mathbf{b_n})$$

Since the coordinate mapping from V to $\mathbb{R}^n$ is linear (Theorem 8 of Section 4.4), we have

$$[T(\mathbf{x})]_{\mathcal{B}} = r_1[T(\mathbf{b_1})]_{\mathcal{B}} + \cdots + r_n[T(\mathbf{b_n})]_{\mathcal{B}}$$

But $\mathcal{B}$-coordinate vectors are in $\mathbb{R}^n$, so we can write this vector equation as a matrix equation; namely,

$[T(\mathbf{x})]_{\mathcal{B}} = M[\mathbf{x}]_{\mathcal{B}}$ where $M = [[T(\mathbf{b_1})]_{\mathcal{B}} \; [T(\mathbf{b_2})]_{\mathcal{B}} \ldots [T(\mathbf{b_n})]_{\mathcal{B}}]$

The matrix M is a matrix representation of T, called the **matrix for T relative to the basis $\mathcal{B}$** and denoted by $[T]_{\mathcal{B}}$

Linear Transformations on $\mathbb{R}^n$

Theorem 8: Diagonal Matrix Representation:

Suppose $A = PDP^{-1}$ where D is a diagonal $n \times n$ matrix. If $\mathcal{B}$ is the basis for $\mathbb{R}^n$ formed from the columns of P, then D is the $\mathcal{B}$ -matrix for the transformation $\mathbf{x} \rightarrow A\mathbf{x}$.

Similarity of Matrix Representations

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\mathbf{x} \arrow[r, "\text{Multiplication}", "\text{by } A"'] \arrow[d, "\text{Multiplication}\\ \text{by } P^{-1}"'] & A\mathbf{x} \\
{[\mathbf{x}]_{\mathcal{B}}} \arrow[r, "\text{Multiplication}", "\text{by } C"'] & {[A\mathbf{x}]_{\mathcal{B}}} \arrow[u, "\text{Multiplication}\\ \text{by } P"']
\end{tikzcd}$$

The set of all matrices similar to a matrix A coincides with the set of all matrix representations of the transformation $\mathbf{x} \to A\mathbf{x}$.