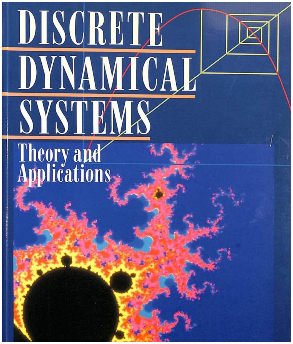
MATH 200C: Linear Algebra



Class 29: Friday, April 24, 2026

- Discrete Dynamical Systems
- Notes on Assignment 26

Eigenvalues and Eigenvectors

Definition: An **eigenvector** of an $n \times n$ matrix A is a nonzero vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$ for some scalar λ.

A scalar λ is called an **eigenvalue** of A if there is a nontrivial solution $\mathbf{x}$ of $A\mathbf{x} = \lambda\mathbf{x}$; such an x is called an eigenvector corresponding to λ.

Diagonalizing A Matrix

$A = \begin{bmatrix} -17 & -30 \\ 10 & 18 \end{bmatrix}$ has eigenvalue 3 with eigenvector $\mathbf{v_1} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$

and eigenvalue -2 with eigenvector $\mathbf{v_2} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

(See Exam 2 or Lecture Notes for Monday, April 13, Class 25)

Form the matrix P whose columns are the eigenvectors:

$P = \begin{bmatrix} -3 & 2 \\ 2 & 1 \end{bmatrix}$. Then P is invertible with $P^{-1} = \begin{bmatrix} 1 & 2 \\ -2 & -3 \end{bmatrix}$

The diagonal matrix is formed from the eigenvalues: $D = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}$

Thus $A = P\,D\,P^{-1}$

Now $A^{1000} = P\,D^{1000}P^{-1} = P \begin{bmatrix} 3^{1000} & 0 \\ 0 & (-2)^{1000} \end{bmatrix} P^{-1}$

Computational Advantage of Diagonalized Matrix

$$A^{1001} = P\,D^{1001}P^{-1} = P\begin{bmatrix} 3^{1001} & 0 \\ 0 & (-2)^{1000} \end{bmatrix} P^{-1}$$

To multiply two 2×2 matrices together takes 8 multiplications and 4 additions so 12 arithmetic procedures.

Right Hand Side requires 24 arithmetic steps

But Left Hand Side $A^{1001} = A \times A \times A \times \cdots \times A$ requires 12,000 steps!

Not Every Matrix is Diagonalizable

$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ has $(A - \lambda I) = \begin{bmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{bmatrix}$ with characteristic polynomial $(1-\lambda)^2$ so $\lambda = 1$ is a root of algebraic multiplicity 2.

Eigenvectors are solutions of $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

But every solution of this system is a scalar multiple of $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Be Careful!

$A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$ has $(A - \lambda I) = \begin{bmatrix} 3-\lambda & 0 \\ 0 & 3-\lambda \end{bmatrix}$ with characteristic polynomial $(3-\lambda)^2$ so $\lambda = 3$ is a root of algebraic multiplicity 2.

Eigenvectors are solutions of $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

But every vector in $\mathbb{R}^n$ is a solution so we can find a linearly independent pair of eigenvectors $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ and A is diagonalizable.

When is a Matrix Diagonalizable?

Theorem 5: The Diagonalization Theorem An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors.

In fact, $A = PDP^{-1}$, with D a diagonal matrix, if and only if the columns of P are n linearly independent eigenvectors of A. In this case, the diagonal entries of D are eigenvalues of A that correspond, respectively, to the eigenvectors in P.

Discrete Dynamical Systems

Suppose $\mathbf{x_0}, \mathbf{x_1}, \mathbf{x_2}, \ldots,$ is a sequence of vectors in $\mathbb{R}^n$ and A is a matrix such that

$$(*)\ \mathbf{x_{k+1}} = A\mathbf{x_k} \text{ for } k = 0, 1, 2, \ldots$$

Then $(*)$ is called a **linear difference equation** or **recurrence equation** or **discrete dynamical system**.

The vector $\mathbf{x_0}$ is called the **initial vector.**

Suppose A is diagonalizable with n linearly independent eigenvectors $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}$ and corresponding eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$.

Assume the eigenvalues are arranged in descending order of magnitude $|\lambda_1| \geq |\lambda_2| \geq \ldots \geq |\lambda_n|$

. Since $\{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}\}$ is a basis for $\mathbb{R}^n$, any initial vector $\mathbf{x_0}$ is a unique linear combination of the eigenvectors $\mathbf{x_0} = c_1\mathbf{v_1} + c_2\mathbf{v_2} + .. + c_n\mathbf{v_n}$ called the **eigenvector decomposition** of $\mathbf{x_0}$.

Theorem: $\mathbf{x_k} = A^k\mathbf{x_0} = c_1(\lambda_1)^k\mathbf{v_1} + c_2(\lambda_2)^k\mathbf{v_2} + \ldots + c_n(\lambda_n)^k\mathbf{v_n}$ $(k = 0, 1, 2, \ldots)$

Theorem:

$$\mathbf{x_k} = A^k \mathbf{x_0} = c_1(\lambda_1)^k \mathbf{v_1} + c_2(\lambda_2)^k \mathbf{v_2} + \ldots + c_n(\lambda_n)^k \mathbf{v_n}$$

$$(k = 0, 1, 2, \ldots)$$

Special Case: Suppose $|\lambda_1| > 1$ and $|\lambda_j| < 1$ for $j = 2, 3, \ldots, n$.

Then for large values of k, we have

$\mathbf{x_{k+1}} \approx \lambda_1 \, \mathbf{x_k}$ and $\mathbf{x_k} \approx \lambda_1^k \, \mathbf{x_1}$

Graphical Description of Solutions in $\mathbb{R}^n$

If A is 2×2, the graph of $\mathbf{x_0}, \mathbf{x_1}, \mathbf{x_2}, \ldots$ is called the **trajectory** of the system.

Case 1: Both eigenvalues are less than 1 in magnitude

The origin is an **attractor**: all trajectories tend toward $\mathbf{0}$

Case 2: Both eigenvalues are greater than 1 in magnitude

The origin is a **repeller**: solutions are unbounded and tend away from origin

Case 3 : One eigenvalue has magnitude greater than 1 and the other eigenvalue has magnitude less than 1.

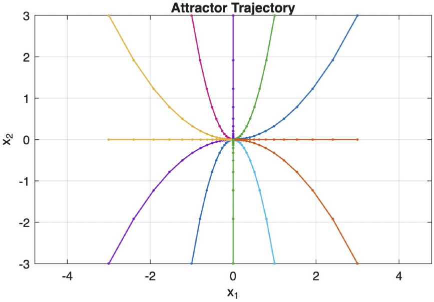
The origin is a **saddle point**: the origin attracts solutions from some directions and repels them in other directions.

Attractor Case

Case 1: Both eigenvalues are less than 1 in magnitude

The origin is an **attractor**: all trajectories tend toward $\mathbf{0}$

$$A = \begin{bmatrix} .8 & 0 \\ 0 & .64 \end{bmatrix}$$

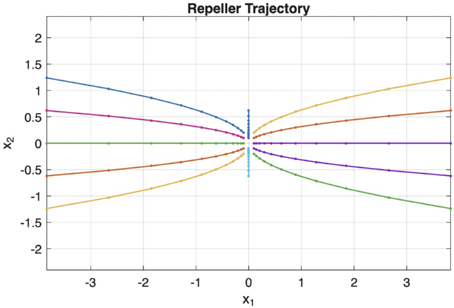


Repeller Case

Case 2: Both eigenvalues are greater than 1 in magnitude

The origin is a **repeller**: solutions are unbounded and tend away from origin

$$A = \begin{bmatrix} 1.44 & 0 \\ 0 & 1.2 \end{bmatrix}$$

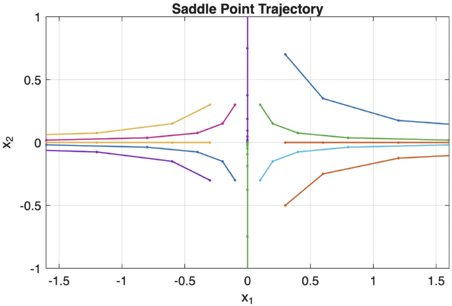


Saddle Point Case

Case 3 : One eigenvalue has magnitude greater than 1 and the other eigenvalue has magnitude less than 1.

The origin is a **saddle point**: the origin attracts solutions from some directions and repels them in other directions.

$$A = \begin{bmatrix} 2.0 & 0 \\ 0 & 0.5 \end{bmatrix}$$



Spiral Toward Origin

$$A = \begin{bmatrix} .8 & .5 \\ -1 & 1 \end{bmatrix}$$