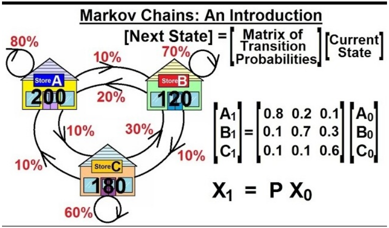
MATH 200C: Linear Algebra



Class 30: Monday, April 27, 2026

- Introduction to Markov Chains
- Notes on Assignment 27

Applications to Markov Chains

Definitions: A vector with **nonnegative** entries that add up to 1 is called a **probability vector**.

Probability Vectors	Not Probability Vectors
$\begin{bmatrix} .2 \\ .7 \\ 0 \\ .1 \end{bmatrix}, \begin{bmatrix} .4 \\ .6 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} .4 \\ .7 \\ 0 \\ -.1 \end{bmatrix}, \begin{bmatrix} .5 \\ .7 \end{bmatrix}, \begin{bmatrix} .3 \\ .3 \\ .3 \end{bmatrix}$

A **stochastic matrix**, also called a **transition matrix** is a square matrix whose columns are probability vectors.

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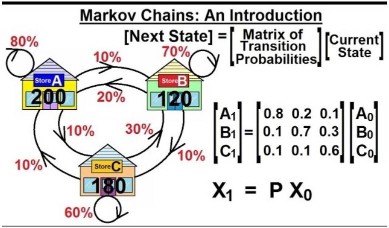
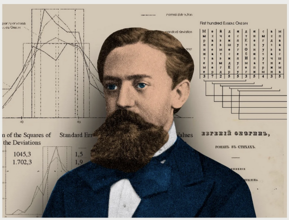
Examples

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \begin{bmatrix} .5 & .5 \\ .5 & .5 \end{bmatrix} \quad \begin{bmatrix} .4 & .7 \\ .6 & .3 \end{bmatrix} \quad \begin{bmatrix} .8 & .2 & .1 \\ .1 & .7 & .3 \\ .1 & .1 & .6 \end{bmatrix}$$





Andrey Andreyevich Markov (June 14, 1856 – July 20, 1922)

A **Markov chain** is a sequence of probability vectors $\mathbf{x_0}, \mathbf{x_1}, \mathbf{x_2}, \ldots,$ together with a stochastic matrix P, such that

$$\mathbf{x_1} = P\mathbf{x_0}, \mathbf{x_2} = P\mathbf{x_1}, \mathbf{x_3} = P\mathbf{x_2}, \ldots$$

Thus the Markov chain is described by the first-order difference equation $\mathbf{x_{k+1}} = P\mathbf{x_k}$ for $k = 0, 1, 2, \ldots$

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$$\mathbf{x_{k+1}} = P\mathbf{x_k} \text{ for } k = 0, 1, 2, \ldots$$

When a Markov chain of vectors in $\mathbb{R}^n$ describes a system or a sequence of experiments, the entries in $\mathbf{x_k}$ list, respectively, the probabilities that the system is in each of n possible states, or the probabilities that the outcome of the experiment is one of n possible outcomes.

For this reason, $\mathbf{x_k}$ is often called a **state vector**.

Predicting the Distant Future

Theorem 10 Stochastic Matrices: If P is a stochastic matrix, then 1 is an eigenvalue of P.

Example: $P = \begin{bmatrix} 4/10 & 7/10 \\ 6/10 & 3/10 \end{bmatrix}$

$$\det\,(P - \lambda I) = \lambda^2 - \tfrac{7}{10}\lambda - \tfrac{3}{10} = (\lambda - 1)(\lambda + \tfrac{3}{10})$$

Steady-State Vectors

If P is a stochastic matrix, then a **steady-state vector** (or **equilibrium vector**) for P is a probability vector $\mathbf{q}$ such that

$$P\mathbf{q} = \mathbf{q}.$$

Example: $P = \begin{bmatrix} 4/10 & 7/10 \\ 6/10 & 3/10 \end{bmatrix}$ and Let $\mathbf{q} = \begin{bmatrix} 7/13 \\ 6/13 \end{bmatrix}$

$$\text{Then } P\mathbf{q} = \begin{bmatrix} 4/10 & 7/10 \\ 6/10 & 3/10 \end{bmatrix} \begin{bmatrix} 7/13 \\ 6/13 \end{bmatrix} = \begin{bmatrix} \frac{28+42}{130} \\ \frac{42+18}{130} \end{bmatrix} = \begin{bmatrix} 7/13 \\ 6/13 \end{bmatrix} = \mathbf{q}$$

Definition: A stochastic matrix is **regular** if some matrix power P^k contains only strictly positive entries.

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Examples

regular: $P = \begin{bmatrix} 4/10 & 7/10 \\ 6/10 & 3/10 \end{bmatrix}$

regular: $P = \begin{bmatrix} 1/2 & 1 \\ 1/2 & 0 \end{bmatrix}$ has $P^2 = \begin{bmatrix} 3/4 & 1/2 \\ 1/4 & 1/2 \end{bmatrix}$

not regular: $P = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ has $P^k = I$ for all $k = 1, 2, \ldots$.

Definition : A sequence of vectors, $\{\mathbf{x_k}\}$ for $k = 1, 2, \ldots,$ **converges** to a vector $\mathbf{q}$ as $k \to \infty$, if the entries in $\mathbf{x_k}$ can be made as close as desired to the corresponding entries in $\mathbf{q}$ by taking k sufficiently large.

Example: $\mathbf{x_k} = \begin{bmatrix} \frac{7}{13} + \left(\frac{3}{10}\right)^k \\ \frac{6}{13} + \left(\frac{-3}{10}\right)^k \end{bmatrix}$ converges to $\mathbf{q} = \begin{bmatrix} \frac{7}{13} \\ \frac{6}{13} \end{bmatrix}$

Theorem 11: If P is an $n \times n$ regular stochastic matrix, then P has a unique steady-state vector $\mathbf{q}$. Further, if $\mathbf{x_0}$ is any initial state and $\mathbf{x_{k+1}} = P\mathbf{x_k}$ for $k = 0, 1, 2, \ldots,$ then the Markov chain $\{\mathbf{x_k}\}$ converges to $\mathbf{q}$ as $k \to \infty$

Example: $P = \begin{bmatrix} 4/10 & 7/10 \\ 6/10 & 3/10 \end{bmatrix}$ has

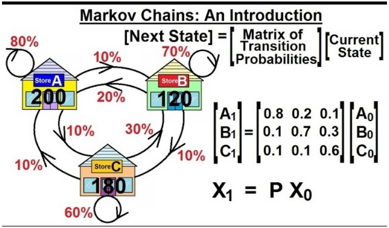
eigenvalue $\lambda_1 = 1$ with eigenvector $\mathbf{v_1} = \begin{bmatrix} 7/13 \\ 6/13 \end{bmatrix}$ and

eigenvalue $\lambda_2 = \frac{-3}{10}$ with eigenvector $\mathbf{v_2} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Any initial $\mathbf{x_0}$ has the form $\begin{bmatrix} a \\ 1-a \end{bmatrix} = 1 \begin{bmatrix} 7/13 \\ 6/13 \end{bmatrix} + \left(\frac{7}{13} - a\right) \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Then $\mathbf{x_k} = 1(1)^k \begin{bmatrix} 7/13 \\ 6/13 \end{bmatrix} + \left(\frac{7}{13} - a\right) \left(\frac{-3}{10}\right)^k \begin{bmatrix} -1 \\ 1 \end{bmatrix} \to \begin{bmatrix} 7/13 \\ 6/13 \end{bmatrix} = \mathbf{q}$

Theorem 11: If P is an $n \times n$ regular stochastic matrix, then P has a unique steady-state vector $\mathbf{q}$. Further, if $\mathbf{x_0}$ is any initial state and $\mathbf{x_{k+1}} = P\mathbf{x_k}$ for $k = 0, 1, 2, \ldots,$ then the Markov chain $\{\mathbf{x_k}\}$ converges to $\mathbf{q}$ as $k \to \infty$



$$A = \begin{bmatrix} 8/10 & 2/10 & 1/10 \\ 1/10 & 7/10 & 3/10 \\ 1/10 & 1/10 & 6/10 \end{bmatrix}$$

Characteristic Polynomial

$$\det\,(A - \lambda I) = (1/10)(\lambda - 1)(5\lambda - 3)(2\lambda - 1)$$

λ_1	Eigenvector	λ_2	Eigenvector	λ_3	Eigenvector
1	$\begin{bmatrix} 9/20 \\ 7/20 \\ 4/20 \end{bmatrix}$	3/5	$\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$	1/2	$\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

$$\mathbf{x_k} \to \mathbf{q} = \begin{bmatrix} 9/20 \\ 7/20 \\ 4/20 \end{bmatrix} = \begin{bmatrix} .45 \\ .35 \\ .20 \end{bmatrix}$$