

# MATH 200C: Linear Algebra



Class 32: Friday, May 1, 2026



- ▶ Notes on Assignment 29



Exam 3 Wednesday Evening  
7 PM – ?  
Focus on Chapters 4 and 5

**Project 2**  
**Age – Class Population Models**  
**DUE: Monday, May 11**

**Final Exam**  
**Thursday, May 14**  
**9 AM – Noon**

## Predicting the Distant Future

**Theorem 10 Stochastic Matrices:** If  $P$  is a stochastic matrix, then 1 is an eigenvalue of  $P$ .

Definition  $\mathbf{q}$  is a **steady-state vector** for a matrix  $P$  if  $P\mathbf{q} = \mathbf{q}$ .

**Theorem 11:** If  $P$  is an  $n \times n$  regular stochastic matrix, then  $P$  has a unique steady-state vector  $\mathbf{q}$ .

Further, if  $\mathbf{x}_0$  is any initial state and  $\mathbf{x}_{k+1} = P\mathbf{x}_k$  for  $k = 0, 1, 2, \dots$ , then the Markov chain  $\{\mathbf{x}_k\}$  converges to  $\mathbf{q}$  as  $k \rightarrow \infty$



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**Example 1 :**  $P = \begin{bmatrix} 4/10 & 7/10 \\ 6/10 & 3/10 \end{bmatrix}$  has

eigenvalue  $\lambda_1 = 1$  with eigenvector  $\mathbf{v}_1 = \begin{bmatrix} 7/13 \\ 6/13 \end{bmatrix}$  and

eigenvalue  $\lambda_2 = \frac{-3}{10}$  with eigenvector  $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Any initial  $\mathbf{x}_0$  has the form  $\begin{bmatrix} a \\ 1-a \end{bmatrix} = 1 \begin{bmatrix} 7/13 \\ 6/13 \end{bmatrix} + \left(\frac{7}{13} - a\right) \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Then  $\mathbf{x}_k = 1(1)^k \begin{bmatrix} 7/13 \\ 6/13 \end{bmatrix} + \left(\frac{7}{13} - a\right) \left(\frac{-3}{10}\right)^k \begin{bmatrix} -1 \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} 7/13 \\ 6/13 \end{bmatrix} = \mathbf{q}$

**Example 2** :  $P = \begin{bmatrix} 2/3 & 1/4 \\ 1/3 & 3/4 \end{bmatrix}$  has

eigenvalue  $\lambda_1 = 1$  with eigenvector  $\mathbf{v}_1 = \begin{bmatrix} 3/7 \\ 4/7 \end{bmatrix}$  and

eigenvalue  $\lambda_2 = \frac{5}{12}$  with eigenvector  $\mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

## A Generic 2 by 2 Stochastic Matrix With All Positive Entries

$$P = \begin{bmatrix} a & 1 - b \\ 1 - a & b \end{bmatrix} \text{ where } 0 < a < 1 \text{ and } 0 < b < 1$$

$P$  has eigenvalue  $\lambda_1 = 1$  with eigenvector

$$\mathbf{v}_1 = \begin{bmatrix} (b - 1)/(a + b - 2) \\ (a - 1)/(a + b - 2) \end{bmatrix} \text{ and}$$

eigenvalue  $\lambda_2 = a + b - 1$  with eigenvector  $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

## Another Interesting Example

$$P = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$$

Characteristic polynomial is  $\lambda^2 - \lambda = \lambda(\lambda - 1)$

Eigenvalues are 1 and 0

Eigenvector for  $\lambda = 1$  is  $\begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$

Eigenvector for  $\lambda = 0$  is  $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

## A 3 by 3 Example

$$A = \begin{bmatrix} 8/10 & 2/10 & 1/10 \\ 1/10 & 7/10 & 3/10 \\ 1/10 & 1/10 & 6/10 \end{bmatrix}$$

Characteristic Polynomial

$$\det(A - \lambda I) = (1/10)(\lambda - 1)(5\lambda - 3)(2\lambda - 1)$$

$\lambda_1$	Eigenvector	$\lambda_2$	Eigenvector	$\lambda_3$	Eigenvector
1	$\begin{bmatrix} 9/20 \\ 7/20 \\ 4/20 \end{bmatrix}$	$3/5$	$\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$	$1/2$	$\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

$$\mathbf{x}_k \rightarrow \mathbf{q} = \begin{bmatrix} 9/20 \\ 7/20 \\ 4/20 \end{bmatrix} = \begin{bmatrix} .45 \\ .35 \\ .20 \end{bmatrix}$$

## Another 3 by 3 Example

$$A = \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}$$

Characteristic Polynomial =  $\det(A - \lambda I) = (\lambda - 1)\lambda^2$

$\lambda_1$	Eigenvector	$\lambda_2$	Eigenvector
1	$\begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix}$	0	$\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

$$\mathbf{x}_k \rightarrow \mathbf{q} = \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix}$$

## Another 3 by 3 Example

$$A = \begin{bmatrix} 1/3 & 1/7 & 3/10 \\ 1/3 & 2/7 & 1/2 \\ 1/3 & 4/7 & 1/5 \end{bmatrix}$$

$$\text{Characteristic Polynomial} = \frac{1}{210}(\lambda - 1)(210\lambda^2 + 38\lambda - 7)$$

$$\text{Eigenvalues: } 1, \frac{-19 + \sqrt{1831}}{210}, \frac{-19 - \sqrt{1831}}{210} \approx (1, 0.11329, -0.294224)$$

$\lambda_1$	Eigenvector	$\lambda_2$	Eigenvector	$\lambda_3$	Eigenvector
1	$\begin{bmatrix} 60/241 \\ 91/241 \\ 90/241 \end{bmatrix}$	0.11329	$\begin{bmatrix} +.82645 \\ -.41541 \\ -.40104 \end{bmatrix}$	-0.294224	$\begin{bmatrix} -.25941 \\ -.54076 \\ +.80017 \end{bmatrix}$

$$\mathbf{x}_k \rightarrow \mathbf{q} = \begin{bmatrix} 60/241 \\ 91/241 \\ 90/241 \end{bmatrix} = \begin{bmatrix} .2490 \\ .3776 \\ .3734 \end{bmatrix}$$

Definition: If  $P$  is a stochastic matrix, then a **steady-state vector** (or **equilibrium vector**) for  $P$  is a probability vector  $\mathbf{q}$  so that that  $P\mathbf{q} = \mathbf{q}$ . If some positive power  $P^k$  of  $P$  contains only strictly positive entries, then  $P$  is called **regular**.

**Theorem:** If  $P$  is a regular  $n \times n$  transition matrix with  $n \geq 2$ , then the following are all true:

- ▶ There is a stochastic matrix  $\Pi = \lim_{m \rightarrow \infty} P^m$
- ▶ Each column of  $\Pi$  is the same probability vector  $\mathbf{q}$ .
- ▶ For any initial probability vector  $\mathbf{x}_0$ , we have  $\lim_{m \rightarrow \infty} P^m \mathbf{x}_0 = \mathbf{q}$ .
- ▶ The vector  $\mathbf{q}$  is the unique probability vector that is an eigenvector of  $P$  associated with the eigenvalue 1.
- ▶ All other eigenvalues  $\lambda$  of  $P$  have  $|\lambda| < 1$ .