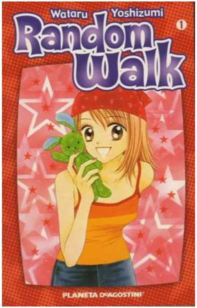
MATH 200C: Linear Algebra



Class 33: Monday, May 4, 2026

- PageRank Algorithm
- Notes on Assignment 30

Exam 3 Wednesday Evening
7 PM – ?
Focus on Chapters 4 and 5

Project 2
Age – Class Population Models
DUE: Monday, May 11

Final Exam
Thursday, May 14
9 AM – Noon

Predicting the Distant Future

Theorem 10 Stochastic Matrices: If P is a stochastic matrix, then 1 is an eigenvalue of P.

Definition $\mathbf{q}$ is a **steady–state vector** for a matrix P if $P\mathbf{q} = \mathbf{q}$.

Theorem 11: If P is an $n \times n$ regular stochastic matrix, then P has a unique steady-state vector $\mathbf{q}$.
Further, if $\mathbf{x_0}$ is any initial state and $\mathbf{x_{k+1}} = P\mathbf{x_k}$ for $k = 0, 1, 2, \ldots,$ then the Markov chain $\{\mathbf{x_k}\}$ converges to $\mathbf{q}$ as $k \to \infty$

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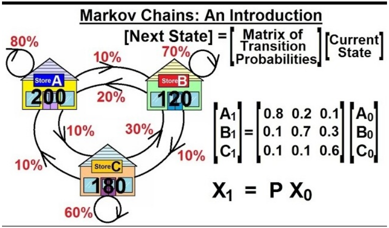

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Example 1 : $P = \begin{bmatrix} 4/10 & 7/10 \\ 6/10 & 3/10 \end{bmatrix}$ has

eigenvalue $\lambda_1 = 1$ with eigenvector $\mathbf{v_1} = \begin{bmatrix} 7/13 \\ 6/13 \end{bmatrix}$ and

eigenvalue $\lambda_2 = \frac{-3}{10}$ with eigenvector $\mathbf{v_2} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Any initial $\mathbf{x_0}$ has the form $\begin{bmatrix} a \\ 1-a \end{bmatrix} = 1 \begin{bmatrix} 7/13 \\ 6/13 \end{bmatrix} + \left(\frac{7}{13} - a\right) \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Then $\mathbf{x_k} = 1(1)^k \begin{bmatrix} 7/13 \\ 6/13 \end{bmatrix} + \left(\frac{7}{13} - a\right) \left(\frac{-3}{10}\right)^k \begin{bmatrix} -1 \\ 1 \end{bmatrix} \to \begin{bmatrix} 7/13 \\ 6/13 \end{bmatrix} = \mathbf{q}$

Example 2 : $P = \begin{bmatrix} 2/3 & 1/4 \\ 1/3 & 3/4 \end{bmatrix}$ has

eigenvalue $\lambda_1 = 1$ with eigenvector $\mathbf{v_1} = \begin{bmatrix} 3/7 \\ 4/7 \end{bmatrix}$ and

eigenvalue $\lambda_2 = \frac{5}{12}$ with eigenvector $\mathbf{v_2} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

A Generic 2 by 2 Stochastic Matrix With All Positive Entries

$$P = \begin{bmatrix} a & 1-b \\ 1-a & b \end{bmatrix} \text{ where } 0 < a < 1 \text{ and } 0 < b < 1$$

P has eigenvalue $\lambda_1 = 1$ with eigenvector

$$\mathbf{v_1} = \begin{bmatrix} \frac{b-1}{a+b-2} \\ \\ \frac{a-1}{a+b-2} \end{bmatrix}$$

and

eigenvalue $\lambda_2 = a + b - 1$ with eigenvector $\mathbf{v_2} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$$P - \lambda I = \begin{bmatrix} a-\lambda & 1-b \\ 1-a & b-\lambda \end{bmatrix}$$

$$\det\,(P - \lambda I) = (a-\lambda)(b-\lambda) - (1-a)(1-b)$$

$$= \lambda^2 - (a+b)\lambda + ab = (\lambda - 1)(\lambda - [a+b-1]).$$

Eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = a + b - 1$

Since a and b are each strictly between 0 and 1, we have $-1 < a + b - 1 < 1$ which means $|\lambda_2| < 1$

To find eigenvector associated with λ_2:

$$P - \lambda_2 I = \begin{bmatrix} a-(a+b-1) & 1-b \\ 1-a & b-(a+b-1) \end{bmatrix} = \begin{bmatrix} 1-b & 1-b \\ 1-a & 1-a \end{bmatrix}$$

which reduces to $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ so components of eigenvector are equal in size and opposite in sign, adding up to 0.

Another Interesting Example

$$P = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$$

Characteristic polynomial is $\lambda^2 - \lambda = \lambda(\lambda - 1)$

Eigenvalues are 1 and 0

Eigenvector for $\lambda = 1$ is $\begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$

Eigenvector for $\lambda = 0$ is $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

A 3 by 3 Example

$$A = \begin{bmatrix} 8/10 & 2/10 & 1/10 \\ 1/10 & 7/10 & 3/10 \\ 1/10 & 1/10 & 6/10 \end{bmatrix}$$

Characteristic Polynomial

$$\det\,(A - \lambda I) = (1/10)(\lambda - 1)(5\lambda - 3)(2\lambda - 1)$$

λ_1	Eigenvector	λ_2	Eigenvector	λ_3	Eigenvector
1	$\begin{bmatrix} 9/20 \\ 7/20 \\ 4/20 \end{bmatrix}$	3/5	$\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$	1/2	$\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

$$\mathbf{x_k} \to \mathbf{q} = \begin{bmatrix} 9/20 \\ 7/20 \\ 4/20 \end{bmatrix} = \begin{bmatrix} .45 \\ .35 \\ .20 \end{bmatrix}$$

Another 3 by 3 Example

$$A = \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}$$

$$\text{Characteristic Polynomial} = \det\,(A - \lambda I) = (\lambda - 1)\lambda^2$$

λ_1	Eigenvector	λ_2	Eigenvector
1	$\begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix}$	0	$\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

$$\mathbf{x_k} \to \mathbf{q} = \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix}$$

Another 3 by 3 Example

$$A = \begin{bmatrix} 1/3 & 1/7 & 3/10 \\ 1/3 & 2/7 & 1/2 \\ 1/3 & 4/7 & 1/5 \end{bmatrix}$$

$$\text{Characteristic Polynomial} = \tfrac{1}{210}(\lambda - 1)(210\lambda^2 + 38\lambda - 7)$$

$$\text{Eigenvalues: } 1, \tfrac{-19+\sqrt{1831}}{210}, \tfrac{-19-\sqrt{1831}}{210} \approx (1, 0.11329, -0.294224)$$

λ_1	Eigenvector	λ_2	Eigenvector	λ_3	Eigenvector
1	$\begin{bmatrix} 60/241 \\ 91/241 \\ 90/241 \end{bmatrix}$	0.11329	$\begin{bmatrix} +.82645 \\ -.41541 \\ -.40104 \end{bmatrix}$	-0.294224	$\begin{bmatrix} -.25941 \\ -.54076 \\ +.80017 \end{bmatrix}$

$$\mathbf{x_k} \to \mathbf{q} = \begin{bmatrix} 60/241 \\ 91/241 \\ 90/241 \end{bmatrix} = \begin{bmatrix} .2490 \\ .3776 \\ .3734 \end{bmatrix}$$

Definition: If P is a stochastic matrix, then a **steady-state vector** (or **equilibrium vector**) for P is a probability vector $\mathbf{q}$ so that that $P\mathbf{q} = \mathbf{q}$. If some positive power P^k of P contains only strictly positive entries, then P is called **regular**.

Theorem: If P is a regular $n \times n$ transition matrix with $n \geq 2$, then the following are all true:

- There is a stochastic matrix $\Pi = \lim_{m \to \infty} P^m$
- Each column of Π is the same probability vector $\mathbf{q}$.
- For any initial probability vector $\mathbf{x_0}$, we have $\lim_{m \to \infty} P^m \mathbf{x_0} = \mathbf{q}$.
- The vector $\mathbf{q}$ is the unique probability vector that is an eigenvector of P associated with the eigenvalue 1.
- All other eigenvalues λ of P have $|\lambda| < 1$.

Let λ be an eigenvalue of an $n \times n$ stochastic matrix P with associated eigenvector $\mathbf{v}$ so that $P\mathbf{v} = \lambda\mathbf{v}$

Let $\mathbf{u}$ be the $1 \times n$ vector each of whose entries is 1.

Then $\mathbf{u}P$ is a $1 \times n$ vector whose entries are the sums of the column entries in P.

But each of these column sums is 1. So $\mathbf{u}P = \mathbf{u}$.

Multiply each side of $P\mathbf{v} = \lambda\mathbf{v}$ on the left by $\mathbf{u}$:

$\mathbf{u}P\mathbf{v} = \mathbf{u}\lambda\mathbf{v}$ so $\mathbf{uv} = \lambda\mathbf{uv}$ and $1\mathbf{uv} - \lambda\mathbf{uv} = 0$

Thus $(1 - \lambda)\mathbf{uv} = 0$. If $\lambda \neq 1$, then $\mathbf{uv} = 0$

But $\mathbf{uv}$ is the sum of entries of $\mathbf{v}$

Theorem: If v is an eigenvector of a stochastic matrix P for an eigenvalue not equal to 1, then the sum of the entries in that eigenvector is 0.

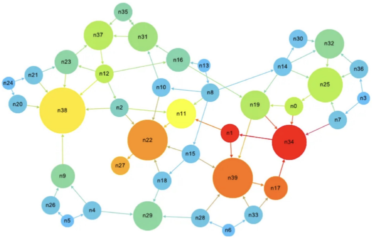
Larry Page and Sergey Brin

Google PageRank Algorithm

Definitions:A **graph** is a collection of points (**vertices**) and lines (**edges**) connecting some of the points.

A **random walk** on a graph is a Markov Chain where at each step the chain is equally likely to move along any of the edges attached to the vertex.

A **directed graph** is a graph in which the vertices are joined not by lines but by arrows.

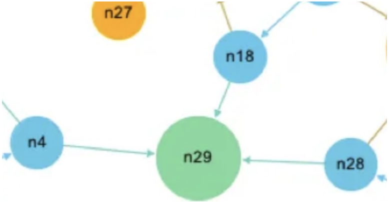


A **simple random walk** on a directed graph allows the chain to move from vertex to vertex but only in the directions allowed by the arrows.

A **dangling node** is a vertex from which no arrow leads out.

Is there a dangling node?

A **dangling node** is a vertex from which no arrow leads out.



Node n29 is dangling.

Google models the Web as a directed graph: vertices are pages and an arrow goes from page j to page i if there is a hyperlink from page j to page i.

The PageRank Algorithm is a simple random walk on this directed graph modified so that the transition matrix is regular.