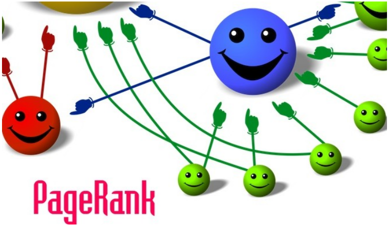
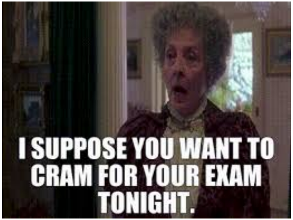
MATH 200C: Linear Algebra



Class 34: Wednesday, May 6, 2026

- PageRank Algorithm



Exam 3 Tonight

7 PM – ?

Focus on Chapters 4 and 5

Last Name	Room
A – K	Warner 105
L – Z	Warner 100

Project 2
Age – Class Population Models
DUE: Monday, May 11

Course Response Forms
In Class
Monday, May 11

Final Exam
Thursday, May 14
9 AM – Noon

Predicting the Distant Future

Theorem 10 Stochastic Matrices: If P is a stochastic matrix, then 1 is an eigenvalue of P.

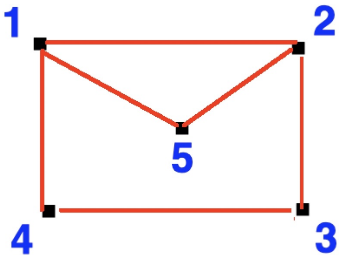
Definition $\mathbf{q}$ is a **steady–state vector** for a matrix P if $P\mathbf{q} = \mathbf{q}$.

Theorem 11: If P is an $n \times n$ regular stochastic matrix, then P has a unique steady-state vector $\mathbf{q}$.
Further, if $\mathbf{x_0}$ is any initial state and $\mathbf{x_{k+1}} = P\mathbf{x_k}$ for $k = 0, 1, 2, \ldots$, then the Markov chain $\{\mathbf{x_k}\}$ converges to $\mathbf{q}$ as $k \to \infty$

Theorem: If P is a regular $n \times n$ transition matrix with $n \geq 2$, then the following are all true:

- There is a stochastic matrix $\Pi = \lim_{m \to \infty} P^m$
- Each column of Π is the same probability vector $\mathbf{q}$.
- For any initial probability vector $\mathbf{x_0}$, we have $\lim_{m \to \infty} P^m \mathbf{x_0} = \mathbf{q}$.
- The vector $\mathbf{q}$ is the unique probability vector that is an eigenvector of P associated with the eigenvalue 1.
- All other eigenvalues λ of P have $|\lambda| < 1$.

Random Walks on Graphs



$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} \textit{One} & \textit{Two} & \textit{Three} & 4 & \textit{Five} \end{array} \\ \begin{bmatrix} 0 & 1/3 & 0 & 1/2 & 1/2 \\ 1/3 & 0 & 1/2 & 0 & 1/2 \\ 0 & 1/3 & 0 & 1/2 & 0 \\ 1/3 & 0 & 1/2 & 0 & 0 \\ 1/3 & 1/3 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

The Transition Matrix P

$$\begin{bmatrix} 0 & \frac{1}{3} & 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{3} & 0 & \frac{1}{2} & 0 \\ \frac{1}{3} & 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 \end{bmatrix}$$

Examine P^2

$$= \begin{bmatrix} \frac{4}{9} & \frac{1}{6} & \frac{5}{12} & 0 & \frac{1}{6} \\ \frac{1}{6} & \frac{4}{9} & 0 & \frac{5}{12} & \frac{1}{6} \\ \frac{5}{18} & 0 & \frac{5}{12} & 0 & \frac{1}{6} \\ 0 & \frac{5}{18} & 0 & \frac{5}{12} & \frac{1}{6} \\ \frac{1}{9} & \frac{1}{9} & \frac{1}{6} & \frac{1}{6} & \frac{1}{3} \end{bmatrix}$$

Examine P^3

$$
\begin{bmatrix}
\frac{1}{9} & \frac{37}{108} & \frac{1}{12} & \frac{31}{72} & \frac{11}{36} \\
\frac{37}{108} & \frac{1}{9} & \frac{31}{72} & \frac{1}{12} & \frac{11}{36} \\
\frac{1}{18} & \frac{31}{108} & 0 & \frac{25}{72} & \frac{5}{36} \\
\frac{31}{108} & \frac{1}{18} & \frac{25}{72} & 0 & \frac{5}{36} \\
\frac{11}{54} & \frac{11}{54} & \frac{5}{36} & \frac{5}{36} & \frac{1}{9}
\end{bmatrix}
$$

Examine P^4

$$
\begin{bmatrix}
\frac{233}{648} & \frac{1}{6} & \frac{167}{432} & \frac{7}{72} & \frac{49}{216} \\
\frac{1}{6} & \frac{233}{648} & \frac{7}{72} & \frac{167}{432} & \frac{49}{216} \\
\frac{167}{648} & \frac{7}{108} & \frac{137}{432} & \frac{1}{36} & \frac{37}{216} \\
\frac{7}{108} & \frac{167}{648} & \frac{1}{36} & \frac{137}{432} & \frac{37}{216} \\
\frac{49}{324} & \frac{49}{324} & \frac{37}{216} & \frac{37}{216} & \frac{11}{54}
\end{bmatrix}
$$

Characteristic Polynomial $= \det(A - \lambda I) = \lambda^5 - \frac{37}{36}\lambda^3 - \frac{1}{9}\lambda^2 + \frac{5}{36}\lambda$
which factors as $\frac{1}{36}\left(\lambda(\lambda-1)(6\lambda+5)(3\lambda-1)(2\lambda+1)\right)$
so eigenvalues are $1, 0, -\frac{5}{6}, \frac{1}{3} - \frac{1}{2}$

Eigenvalues and their corresponding eigenvectors are:

1	0	-5/6	1/3	-1/2
$\begin{bmatrix} 1/4 \\ 1/4 \\ 1/6 \\ 1/6 \\ 1/6 \end{bmatrix}$	$\begin{bmatrix} 3/2 \\ -3/2 \\ -1 \\ 1 \\ 0 \end{bmatrix}$	$\begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 1/2 \\ 1/2 \\ -1 \\ -1 \\ 1 \end{bmatrix}$	$\begin{bmatrix} -3/4 \\ -3/4 \\ 1/4 \\ 1/4 \\ 1 \end{bmatrix}$

$$\text{Steady State Vector is } \mathbf{q} = \begin{bmatrix} 1/4 \\ 1/4 \\ 1/6 \\ 1/6 \\ 1/6 \end{bmatrix} \approx \begin{bmatrix} .25 \\ .25 \\ .16666 \\ 16666 \\ 16666 \end{bmatrix}$$

$$P^{32} =$$

$$\begin{bmatrix} 0.2505851001 & 0.2494149000 & 0.2508776500 & 0.2491223499 & 0.2499999999 \\ 0.2494149000 & 0.2505851001 & 0.2491223499 & 0.2508776500 & 0.2499999999 \\ 0.1672517667 & 0.1660815666 & 0.1675443167 & 0.1657890166 & 0.1666666667 \\ 0.1660815666 & 0.1672517667 & 0.1657890166 & 0.1675443167 & 0.1666666667 \\ 0.1666666666 & 0.1666666666 & 0.1666666667 & 0.1666666667 & 0.1666666668 \end{bmatrix}$$

Theorem: If P is a regular $n \times n$ transition matrix with $n \geq 2$, then the following are all true:

- There is a stochastic matrix $\Pi = \lim_{m\to\infty} P^m$
- Each column of Π is the same probability vector $\mathbf{q}$.
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- The vector $\mathbf{q}$ is the unique probability vector that is an eigenvector of P associated with the eigenvalue 1.
- All other eigenvalues λ of P have $|\lambda| < 1$.

x_0	x_1	x_2	x_3
1	0	4/9	7/36
0	1/3	1/6	7/27
0	0	1/9	1/18
0	1/3	1/6	31/108
0	1/3	1/9	11/54

Larry Page and Sergey Brin

Google PageRank Algorithm

Definitions:A **graph** is a collection of points (**vertices**) and lines (**edges**) connecting some of the points.

A **random walk** on a graph is a Markov Chain where at each step the chain is equally likely to move along any of the edges attached to the vertex.

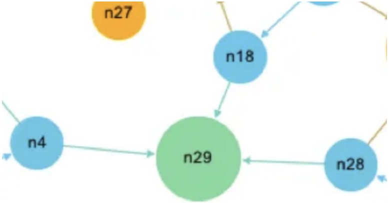
A **directed graph** is a graph in which the vertices are joined not by lines but by arrows.

A **simple random walk** on a directed graph allows the chain to move from vertex to vertex but only in the directions allowed by the arrows.

A **dangling node** is a vertex from which no arrow leads out.

Is there a dangling node?

A **dangling node** is a vertex from which no arrow leads out.



Node n29 is dangling.

Google models the Web as a directed graph: vertices are pages and an arrow goes from page j to page i if there is a hyperlink from page j to page i.

The PageRank Algorithm is a simple random walk on this directed graph modified so that the transition matrix is regular.

Definition: If P is a stochastic matrix, then a **steady-state vector** (or **equilibrium vector**) for P is a probability vector $\mathbf{q}$ so that that $P\mathbf{q} = \mathbf{q}$. If some positive power P^k of P contains only strictly positive entries, then P is called **regular**.

Theorem: If P is a regular $n \times n$ transition matrix with $n \geq 2$, then the following are all true:

- There is a stochastic matrix $\Pi = \lim_{m\to\infty} P^m$
- Each column of Π is the same probability vector $\mathbf{q}$.
- For any initial probability vector $\mathbf{x_0}$, we have $\lim_{m\to\infty} P^m\mathbf{x_0} = \mathbf{q}$.
- The vector $\mathbf{q}$ is the unique probability vector that is an eigenvector of P associated with the eigenvalue 1.
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Google models the Web as a directed graph: vertices are pages and an arrow goes from page j to page i if there is a hyperlink from page j to page i.

The PageRank Algorithm is a simple random walk on this directed graph modified so that the transition matrix is regular.

Adjustment 1: If the surfer reaches a dangling node, then the surfer picks any web page with equal probability and moves to that page:

If j is an absorbing state for P an $n \times n$ matrix , then replace column j of P with the vector all of whose entries are $1/n$ to create a new matrix P^*

Adjustment 2: Let p be a number between 0 and 1.

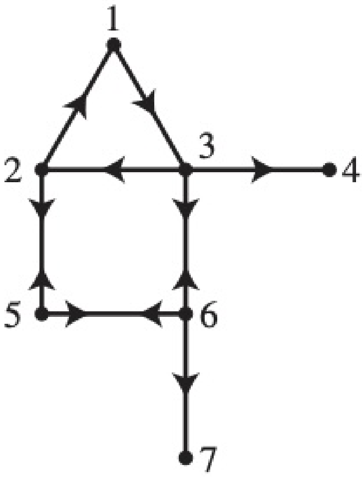
Assume the surfer is now at page j. With probability p, the surfer picks from among all possible links from page with equal probability and moves to that page.

With probability $1 - p$, the surfer picks any page in the Web with equal probability and moves to that page; that is, the **Google matrix** is the new transition matrix G where

$$G = pP^* + (1–p)K$$

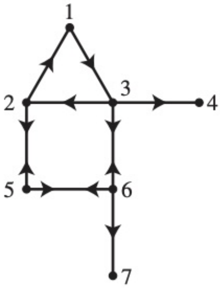
where K is an $n \times n$ matrix all of whose entries are $1/n$.

Example: A Small Web



Nodes 4 and 7 are Dangling.
They are Absorbing States.

Example: Transition Matrix For A Small Web



$$P = \begin{array}{c} \\ \\ \end{array}\overset{\begin{array}{ccccccc}1 & 2 & 3 & 4 & 5 & 6 & 7\end{array}}{\begin{bmatrix} 0 & 1/2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/3 & 0 & 1/2 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1/3 & 0 \\ 0 & 0 & 1/3 & 1 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 & 0 & 1/3 & 0 \\ 0 & 0 & 1/3 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/3 & 1 \end{bmatrix}} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array}$$

Adjustment 1: If the surfer reaches a dangling node, then the surfer picks any web page with equal probability and moves to that page:

$$P = \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{matrix} \\ \begin{bmatrix} 0 & 1/2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/3 & 0 & 1/2 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1/3 & 0 \\ 0 & 0 & 1/3 & 1 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 & 0 & 1/3 & 0 \\ 0 & 0 & 1/3 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/3 & 1 \end{bmatrix} \end{array} \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{matrix}$$

$$P_* = \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{matrix} \\ \begin{bmatrix} 0 & 1/2 & 0 & 1/7 & 0 & 0 & 1/7 \\ 0 & 0 & 1/3 & 1/7 & 1/2 & 0 & 1/7 \\ 1 & 0 & 0 & 1/7 & 0 & 1/3 & 1/7 \\ 0 & 0 & 1/3 & 1/7 & 0 & 0 & 1/7 \\ 0 & 1/2 & 0 & 1/7 & 0 & 1/3 & 1/7 \\ 0 & 0 & 1/3 & 1/7 & 1/2 & 0 & 1/7 \\ 0 & 0 & 0 & 1/7 & 0 & 1/3 & 1/7 \end{bmatrix} \end{array} \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{matrix}$$

For Adjustment 2: Let K be the $n \times n$ matrix all of whose entries are $1/n$.

Here $n = 7$

$$
\begin{array}{c}
\begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\
P_* = \begin{bmatrix}
0 & 1/2 & 0 & 1/7 & 0 & 0 & 1/7 \\
0 & 0 & 1/3 & 1/7 & 1/2 & 0 & 1/7 \\
1 & 0 & 0 & 1/7 & 0 & 1/3 & 1/7 \\
0 & 0 & 1/3 & 1/7 & 0 & 0 & 1/7 \\
0 & 1/2 & 0 & 1/7 & 0 & 1/3 & 1/7 \\
0 & 0 & 1/3 & 1/7 & 1/2 & 0 & 1/7 \\
0 & 0 & 0 & 1/7 & 0 & 1/3 & 1/7
\end{bmatrix}
\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array}
\end{array}
$$

$$
K = \begin{bmatrix}
1/7 & 1/7 & 1/7 & 1/7 & 1/7 & 1/7 & 1/7 \\
1/7 & 1/7 & 1/7 & 1/7 & 1/7 & 1/7 & 1/7 \\
1/7 & 1/7 & 1/7 & 1/7 & 1/7 & 1/7 & 1/7 \\
1/7 & 1/7 & 1/7 & 1/7 & 1/7 & 1/7 & 1/7 \\
1/7 & 1/7 & 1/7 & 1/7 & 1/7 & 1/7 & 1/7 \\
1/7 & 1/7 & 1/7 & 1/7 & 1/7 & 1/7 & 1/7 \\
1/7 & 1/7 & 1/7 & 1/7 & 1/7 & 1/7 & 1/7
\end{bmatrix}
$$

Adjustment 2: Let p be a number between 0 and 1. Assume the surfer is now at page j. With probability p, the surfer picks from among all possible links from page with equal probability and moves to that page. With probability $1 - p$, the surfer picks any page in the Web with equal probability and moves to that page; that is, the **Google matrix** is the new transition matrix G where

$$G = pP^* + (1{-}p)K$$

where K is an $n \times n$ matrix all of whose entries are $1/n$.

With $p = .85$, $G = .85P^* + .15K$:

$$\begin{bmatrix} .021429 & .446429 & .021429 & .142857 & .021429 & .021429 & .142857 \\ .021429 & .021429 & .304762 & .142857 & .446429 & .021429 & .142857 \\ .871429 & .021429 & .021429 & .142857 & .021429 & .304762 & .142857 \\ .021429 & .021429 & .304762 & .142857 & .021429 & .021429 & .142857 \\ .021429 & .446429 & .021429 & .142857 & .021429 & .304762 & .142857 \\ .021429 & .021429 & .304762 & .142857 & .446429 & .021429 & .142857 \\ .021429 & .021429 & .021429 & .142857 & .021429 & .304762 & .142857 \end{bmatrix}$$

The 20th Power of the Google Matrix G^20 looks like:

$$\begin{bmatrix} 0.1163 & 0.1163 & 0.1163 & 0.1163 & 0.1163 & 0.1163 & 0.1163 \\ 0.1686 & 0.1686 & 0.1686 & 0.1686 & 0.1686 & 0.1686 & 0.1686 \\ 0.1913 & 0.1913 & 0.1913 & 0.1913 & 0.1913 & 0.1913 & 0.1913 \\ 0.0988 & 0.0988 & 0.0988 & 0.0988 & 0.0988 & 0.0988 & 0.0988 \\ 0.1641 & 0.1641 & 0.1641 & 0.1641 & 0.1641 & 0.1641 & 0.1641 \\ 0.1686 & 0.1686 & 0.1686 & 0.1686 & 0.1686 & 0.1686 & 0.1686 \\ 0.0924 & 0.0924 & 0.0924 & 0.0924 & 0.0924 & 0.0924 & 0.0924 \end{bmatrix}$$

The Steady-State Vector is

$$\mathbf{q} = \begin{bmatrix} .116293 \\ .168567 \\ .191263 \\ .098844 \\ .164054 \\ .168567 \\ .092413 \end{bmatrix}$$

Steady State $\mathbf{q}$	WebSite	Sorted	PageRank
.116293	**1**	.191263	**3**
.168567	**2**	.168567	**2 (tied with 6)**
.191263	**3**	.168567	**6 (tied with 2**
.098844	**4**	.164054	**5**
.164054	**5**	.116293	1
.168567	**6**	.098844	4
.092413	**7**	.092413	7