MATH 200C: Linear Algebra

Question: Luigi, Daisy and Bowser are playing catch. Luigi always throws the ball to Daisy and Daisy always throws the ball to Bowser. Bowser throws the ball to Luigi 2/5 of the time and to Daisy 3/5 of the time. In the long run, what percentage of the time does each character receive the ball?

Class 35: Friday, May 8, 2026

- Notes on Assignment 31
- Notes on Exam 3

Project 2
Age – Class Population Models
DUE: Monday, May 11

Course Response Forms
In Class
Monday, May 11
Bring Internet Compatible Device

Final Exam
Thursday, May 14
9 AM – Noon
Warner 100

Theorem 11: If P is an $n \times n$ regular stochastic matrix, then P has a unique steady-state vector $\mathbf{q}$.
Further, if $\mathbf{x_0}$ is any initial state and $\mathbf{x_{k+1}} = P\mathbf{x_k}$ for $k = 0, 1, 2, \ldots$, then the Markov chain $\{\mathbf{x_k}\}$ converges to $\mathbf{q}$ as $k \to \infty$

Theorem: If P is a regular $n \times n$ transition matrix with $n \geq 2$, then the following are all true:

- There is a stochastic matrix $\Pi = \lim_{m\to\infty} P^m$
- Each column of Π is the same probability vector $\mathbf{q}$.
- For any initial probability vector $\mathbf{x_0}$, we have $\lim_{m\to\infty} P^m \mathbf{x_0} = \mathbf{q}$.
- The vector $\mathbf{q}$ is the unique probability vector that is an eigenvector of P associated with the eigenvalue 1.
- All other eigenvalues λ of P have $|\lambda| < 1$.

What Have We Proved So Far?

If P is a regular $n \times n$ stochastic matrix, then

- There will always be an eigenvalue λ_1 with value 1
- If P has n distinct eigenavlues $\lambda_1 = 1, \lambda_2, \ldots, \lambda_n$ with corresponding eigenvectors $\mathbf{x_1}, \mathbf{x_1}, \ldots, \mathbf{x_n}$, then the set of eigenvalues is linearly independent and spans $\mathbb{R}^n$
- For each possible starting vector $\mathbf{x_0}$, there are constants $c_1, c_2, \ldots, c_n$ such that $c_1\mathbf{x_1} + c_2\mathbf{x_2} + c_3\mathbf{x_3} + \cdots + c_n\mathbf{x_n}$
- $\mathbf{x_{k+1}} = P\mathbf{x_k} =$
  $c_1(\lambda_1)^k\mathbf{x_1} + c_2(\lambda_2)^k\mathbf{x_2} + c_3(\lambda_3)^k\mathbf{x_3} + \cdots + c_n(\lambda_n)^k\mathbf{x_n}$
  $= c_1\mathbf{x_1} + c_2(\lambda_2)^k\mathbf{x_2} + c_3(\lambda_3)^k\mathbf{x_3} + \cdots + c_n(\lambda_n)^k\mathbf{x_n}$
- If $\lambda \neq 1$ is an eigenvalue of P with corresponding eigenvector $\mathbf{x}$, then the sum of the coordinates of that eigenvector will be 0,

Our Goals

- If $\lambda \neq 1$ is an eigenvalue of P, then $|\lambda| < 1$
- If $\mathbf{v}$ is an eigenvector associated with the eigenvalue $\lambda = 1$, then all components of $\mathbf{v}$ have the same sign.
- The space of eigenvectors associated with eigenvalue 1 for a stochastic matrix has dimension 1.

Theorem A: If $\lambda \neq 1$ is an eigenvalue of P, then $|\lambda| < 1$

Proof: Recall first that P^T has the same eigenvalues as P and that the row sums of P^T are all 1.

Suppose $\mathbf{w}$ is an eigenvector of P^T associated with λ.

Thus $P\mathbf{w} = \lambda\mathbf{w}$

Let w_{max} be the largest component of $\mathbf{w}$.

This entry occurs in some position, say the jth, of the vector $\mathbf{w}$.

Let $a_1, a_2, \ldots, a_j, \ldots, a_n$ be the elements in the jth row of P.

We' will now examine the jth component of $P\mathbf{w}$.

By the rules of matrix multiplication, that component will be

$$a_1 w_1 + a_2 w_2 + a_3 w_3 + \cdots + a_j w_j + \ldots a_n w_n$$

Now $w_1 \leq w_j, w_2 \leq w_j, \ldots, w_j = w_j, \ldots, w_n \leq w_j$

Then $a_1 w_1 + a_2 w_2 + a_3 w_3 + \cdots + a_j w_j + \ldots a_n w_n$

$$\leq a_1 w_j + a_2 w_j + a_3 w_j + \ldots a_n w_j = (a_1 + a_2 + \cdots + a_n) w_j = w_j$$

Thus the jth component of $P\mathbf{w}$ is at most w_j

On the other hand, if $\lambda > 1$, then the jth of $\lambda\mathbf{w}$ will be greater than w_j, which contradicts the fact that $P\mathbf{w} = \lambda\mathbf{w}$.

A similar argument (you supply the details) shows that λ can't be smaller than -1; that is $-1 < \lambda < 1$ which means $|\lambda| < 1$.

The Triangle Inequality

The Absolute Value of a sum of real numbers is less than or equal to the sum of the absolute values

$$|\sum_i y_i| \leq \sum_i |y_i|$$

with strict inequality when the y_i are of mixed sign.

Example: $|3+6-5| = |4| = 4 < |3|+|6|+|-5| = 3+6+5 = 14.$

Theorem B If $\mathbf{v}$ is an eigenvector associated with the eigenvalue $\lambda = 1$, then all components of $\mathbf{v}$ have the same sign.

Proof: We will use proof by contradiction. Suppose some entries of $\mathbf{v}$ are positive and some are negative with

$$\begin{bmatrix} v_1 \\ v_2 \\ \dots \\ v_n \end{bmatrix}.$$

Examine the ith component v_i of $\mathbf{v}$. From $P\mathbf{v} = \mathbf{v}$, we have v_i is the product of the ith row of P and the column vector $\mathbf{v}$; that is, $v_i = P_{i1}v_1 + P_{i2}v_2 + \cdots + P_{in}v_n$. In summation terms,

$$v_i = \sum_{j=1}^{n} P_{ij} v_j$$

] Now some of the terms being added are positive and some are negative since all the P_{ij} are positive. Thus we have

$$(*)\ |v_i| = |\sum_{j=1}^{n} P_{ij} v_j| < \sum_{j=1}^{n} P_{ij} |v_j|$$

$$(*)\ |v_i| = |\sum_{j=1}^{n} P_{ij} v_j| < \sum_{j=1}^{n} P_{ij} |v_j|$$

Sum both sides of the inequality in (*) from $i = 1$ to $i = n$ and swap the i and j summations. Then use the fact that all column sums in P are 1 to find

$$\sum_{i=1}^{n} |v_i| < \sum_{i=1}^{n} \sum_{j=1}^{n} P_{ij} |v_j| = \sum_{j=1}^{n} \left(\sum_{i=1}^{n} P_{ij} \right) |v_j| = \sum_{j=1}^{n} |v_j|$$

which is a contradiction.

If $v_i \geq 0$ and not all v_i are zero, then $v_i > 0$ follows from $v_i = \sum_{j=1}^{n} P_{ij} v_j$ and $P_{ij} >$.

Similarly, $v_i \leq 0$ for all i implies that each $v_i < 0$.

Theorem C: If $\{\mathbf{v}, \mathbf{w}\}$ is a linearly independent set of vectors in $\mathbb{R}^n$, then we can create a vector $\mathbf{x} = s\mathbf{v} + t\mathbf{w}$ (a linear combination of $\mathbf{v}$ and $\mathbf{w}$ where s and t are not both 0 so that $\mathbf{x}$ has both positive and negative components.

Proof:. Let d be the sum of the components of $\mathbf{v}$

We need to consider two cases:

Case 1: If $d = 0$, then $\mathbf{v}$ must have some positive and some negative components. We can let $s = 1$ and $t = 0$.

Case 2: If $d \neq 0$, set $s = \frac{\sum_i w_i}{d}$ and $t = 1$. Now neither s nor t is 0 so $\mathbf{x} = s\mathbf{v} + t\mathbf{w}$ is not the zero vector as $\{\mathbf{v}, \mathbf{w}\}$ is linearly independent. But the sum of the components of $\mathbf{x}$ is 0 and thus it must contain both positive and negative components.

Theorem D : The space of eigenvectors associated with eigenvalue 1 for a stochastic matrix has dimension 1.

Proof: Suppose, to the contrary, that this space contains a pair $\{\mathbf{v}, \mathbf{w}\}$ of linearly independent vectors.

Use Theorem C to create a vector $\mathbf{x} = s\mathbf{v} + t\mathbf{w}$ which has both positive and negative components.

But $\mathbf{x} = s\mathbf{v} + t\mathbf{w}$ is a linear combination of two eigenvectors so it is also an eigenvector associated with eigenvalue 1
($P\mathbf{x} = P(s\mathbf{v} + t\mathbf{w}) = sP\mathbf{v} + tP\mathbf{w} = s\mathbf{v} + t\mathbf{w} = \mathbf{x}$)

This contradicts Theorem B so we can not have such a pair of linearly independent eigenvectors.