MATH 200C: Linear Algebra

Class 36: Monday, May 11, 2026

- Notes on Assignment 32
- Course Response Forms

Project 2
Age – Class Population Models: DUE: Today

Course Response Forms
In Class Today

Additional Office Hours This Week
Wednesday, 9 – 11 AM

Final Exam
Thursday, May 14
9 AM – Noon
Warner 100

Theorem 11: If P is an $n \times n$ regular stochastic matrix, then P has a unique steady-state vector $\mathbf{q}$.

Further, if $\mathbf{x_0}$ is any initial state and $\mathbf{x_{k+1}} = P\mathbf{x_k}$ for $k = 0, 1, 2, \ldots,$ then the Markov chain $\{\mathbf{x_k}\}$ converges to $\mathbf{q}$ as $k \to \infty$

Theorem: If P is a regular $n \times n$ transition matrix with $n \geq 2$, then the following are all true:

- There is a stochastic matrix $\Pi = \lim_{m\to\infty} P^m$
- Each column of Π is the same probability vector $\mathbf{q}$.
- For any initial probability vector $\mathbf{x_0}$, we have $\lim_{m\to\infty} P^m \mathbf{x_0} = \mathbf{q}$.
- The vector $\mathbf{q}$ is the unique probability vector that is an eigenvector of P associated with the eigenvalue 1.
- All other eigenvalues λ of P have $|\lambda| < 1$.

What Have We Proved So Far?

If P is a regular $n \times n$ stochastic matrix, then

- There will always be an eigenvalue λ_1 with value 1
- If P has n distinct eigenavlues $\lambda_1 = 1, \lambda_2, \dots, \lambda_n$ with corresponding eigenvectors $\mathbf{x_1}, \mathbf{x_1}, \dots, \mathbf{x_n}$, then the set of eigenvalues is linearly independent and spans $\mathbb{R}^n$
- For each possible starting vector $\mathbf{x_0}$, there are constants $c_1, c_2, \dots, c_n$ such that $c_1\mathbf{x_1} + c_2\mathbf{x_2} + c_3\mathbf{x_3} + \cdots + c_n\mathbf{x_n}$
- $\mathbf{x_{k+1}} = P\mathbf{x_k} =$
  $c_1(\lambda_1)^k\mathbf{x_1} + c_2(\lambda_2)^k\mathbf{x_2} + c_3(\lambda_3)^k\mathbf{x_3} + \cdots + c_n(\lambda_n)^k\mathbf{x_n}$
  $= c_1\mathbf{x_1} + c_2(\lambda_2)^k\mathbf{x_2} + c_3(\lambda_3)^k\mathbf{x_3} + \cdots + c_n(\lambda_n)^k\mathbf{x_n}$
- If $\lambda \neq 1$ is an eigenvalue of P with corresponding eigenvector $\mathbf{x}$, then the sum of the coordinates of that eigenvector will be 0,
- If $\lambda \neq 1$ is an eigenvalue of P, then $|\lambda| < 1$
- **The coordinates of any eigenvalue associated with $\lambda = 1$ all have the same sign**

Our Goal Today

- The space of eigenvectors associated with eigenvalue 1 for a stochastic matrix has dimension 1.

Theorem C: If $\{\mathbf{v}, \mathbf{w}\}$ is a linearly independent set of vectors in $\mathbb{R}^n$, then we can create a vector $\mathbf{x} = s\mathbf{v} + t\mathbf{w}$ (a linear combination of $\mathbf{v}$ and $\mathbf{w}$ where s and t are not both 0 so that $\mathbf{x}$ has both positive and negative components.

Proof:. Let d be the sum of the components of $\mathbf{v}$

We need to consider two cases:

Case 1: If $d = 0$, then $\mathbf{v}$ must have some positive and some negative components. We can let $s = 1$ and $t = 0$.

Case 2: If $d \neq 0$, set $s = \frac{\sum_i w_i}{d}$ and $t = 1$. Now neither s nor t is 0 so $\mathbf{x} = s\mathbf{v} + t\mathbf{w}$ is not the zero vector as $\{\mathbf{v}, \mathbf{w}\}$ is linearly independent. But the sum of the components of $\mathbf{x}$ is 0 and thus it must contain both positive and negative components.

Theorem D : The space of eigenvectors associated with eigenvalue 1 for a stochastic matrix has dimension 1.

Proof: Suppose, to the contrary, that this space contains a pair $\{\mathbf{v}, \mathbf{w}\}$ of linearly independent vectors.

Use Theorem C to create a vector $\mathbf{x} = s\mathbf{v} + t\mathbf{w}$ which has both positive and negative components.

But $\mathbf{x} = s\mathbf{v} + t\mathbf{w}$ is a linear combination of two eigenvectors so it is also an eigenvector associated with eigenvalue 1 ($P\mathbf{x} = P(s\mathbf{v} + t\mathbf{w}) = sP\mathbf{v} + tP\mathbf{w} = s\mathbf{v} + t\mathbf{w} = \mathbf{x}$)

This contradicts Theorem B so we can not have such a pair of linearly independent eigenvectors.

Let's Do Course Response Forms

https://go.middlebury.edu/crf

The Final Exam

Date/Time	Thursday, 9 AM to Noon
Location	Warner 100
Calculators/Computers/Text Book?	No
"Cheat Sheet? "	Yes
Format	Answer 4 of 5 Questions


GOOD
LUCK ON
FINALS!!