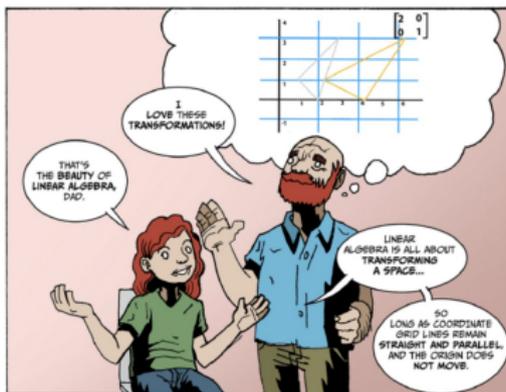


# MATH 200C: Linear Algebra



Class 9: February 27, 2026



- ▶ Notes on Assignment 7
- ▶ Linear Transformations

# Announcements

Exam 1: Next Wednesday, 7 PM -  
No Time Limit

No Books, Computers, Smart Phones,  
etc.

**One Page of Your Own Notes  
OK**

## Highlights from Section 1.7: Linear Independence

Definition An indexed set of vectors  $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p\}$  is **linearly independent** if the vector equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \dots + x_p\mathbf{v}_p = \mathbf{0} \text{ has only the } \mathbf{trivial} \text{ solution}$$
$$x_1 = 0, x_2 = 0, \dots, x_p = 0$$

**Linear Independence** means

- ▶ There is no **interesting** way of writing  $\mathbf{0}$  as a linear combination of the  $\mathbf{v}$ 's.
- ▶ The vector-matrix equation  $A\mathbf{x} = \mathbf{0}$  has the unique solution  $\mathbf{x} = \mathbf{0}$  where  $A$  is the matrix whose columns are the  $\mathbf{v}$ 's.

The set  $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p\}$  is linearly independent if the equation  $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \dots + x_p\mathbf{v}_p = \mathbf{0}$  has only the trivial solution  $x_1 = 0, x_2 = 0, \dots, x_p = 0$

### Example

The set of vectors  $\left\{ \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 10 \end{bmatrix} \right\}$  is linearly independent

$$\begin{bmatrix} 1 & 2 & 3 & | & 0 \\ 4 & 5 & 6 & | & 0 \\ 7 & 8 & 10 & | & 0 \end{bmatrix} \text{ reduces to } \begin{bmatrix} 1 & 0 & 0 & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & 0 & 1 & | & 0 \end{bmatrix}$$

The set  $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p\}$  is **linearly dependent** if there exist weights  $c_1, c_2, \dots, c_p$  **not all zero** such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \dots + c_p\mathbf{v}_p = \mathbf{0}$$

The set  $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p\}$  is **linearly dependent** if there exist weights  $c_1, c_2, \dots, c_p$  **not all zero** such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \dots + c_p\mathbf{v}_p = \mathbf{0}$$

The set of vectors  $\left\{ \begin{bmatrix} 5 \\ 9 \\ 1 \end{bmatrix}, \begin{bmatrix} 11 \\ 25 \\ -9 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 6 \end{bmatrix} \right\}$  is linearly dependent.

$$\left[ \begin{array}{ccc|c} 5 & 11 & 2 & 0 \\ 9 & 25 & 1 & 0 \\ 1 & -9 & 6 & 0 \end{array} \right] \text{ reduces to } \left[ \begin{array}{ccc|c} 1 & 0 & \frac{3}{2} & 0 \\ 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

Thus  $c_1 = (-3/2)c_3$ ,  $c_2 = (1/2)c_3$ ,  $c_3 =$  free choice

One simple choice is  $c_1 = -3$ ,  $c_2 = 1$ ,  $c_3 = 2$

With that simple choice:  $c_1 = -3, c_2 = 1, c_3 = 2$ , we have for

$$\left\{ \mathbf{v}_1 = \begin{bmatrix} 5 \\ 9 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 11 \\ 25 \\ -9 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 2 \\ 1 \\ 6 \end{bmatrix} \right\}$$

$$-3\mathbf{v}_1 + 1\mathbf{v}_2 + 2\mathbf{v}_3 = \mathbf{0}$$

$$\text{so } \mathbf{v}_2 = 3\mathbf{v}_1 - 2\mathbf{v}_3$$

( *Exercise:* Write  $\mathbf{v}_3$  as a linear combination of  $\mathbf{v}_1$  and  $\mathbf{v}_3$ ) Since  $\mathbf{v}_2$

is a linear combination  $\mathbf{v}_1$  and  $\mathbf{v}_3$ , we can rewrite any linear combination of  $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$  as a linear combination of  $\{\mathbf{v}_1, \mathbf{v}_3\}$

$$\text{Thus } \text{Span} (\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}) = \text{Span} (\{\mathbf{v}_1, \mathbf{v}_3\})$$

## Observations

- ▶ The columns of a matrix  $A$  are linearly independent if and only if the equation  $A\mathbf{x} = \mathbf{0}$  has only the trivial solution.
- ▶ A set of exactly one vector is linearly independent if and only if the vector is not the zero vector.
- ▶ A set of exactly two vectors is linearly independent if and only if neither vector is a nonzero scalar multiple of the other vector.

**Theorem 7 (Characterization of Linearly Dependent Sets) :**

An indexed set of two or more vectors  $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p\}$  in  $\mathbb{R}^n$  is linearly dependent if at least one of the vectors is a linear combination of the other vectors.

**Theorem 8** : If a set contains more vectors than there are entries in each vector, then the set is linearly dependent; any set of vectors  $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p\}$  in  $\mathbb{R}^n$  is linearly dependent if  $p > n$ .

**Theorem 8** : If a set contains more vectors than there are entries in each vector, then the set is linearly dependent; any set of vectors  $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p\}$  in  $\mathbb{R}^n$  is linearly dependent if  $p > n$ .

If  $p \leq n$ , there is no guarantee that the set is linearly independent.

**Theorem 9:** If a set of vectors contains the zero vector, then the set is linearly dependent.

**There are sets of vectors, all nonzero, which are linearly dependent.**

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## Highlights from Section 1.8: Linear Transformations

Think of  $A\mathbf{x} = \mathbf{b}$  as “Matrix  $A$  acting on vector  $\mathbf{x}$  to produce a vector  $\mathbf{b}$ ”

Or “Multiply  $A$  by  $\mathbf{x}$  to get a new product  $\mathbf{b}$ ”

If  $A$  is  $m \times n$  and  $\mathbf{x}$  is  $n \times 1$  then  $\mathbf{b} = A\mathbf{x}$  is  $m \times 1$ .

Recall: The vector  $\mathbf{b} = A\mathbf{x}$  is a linear combination of the columns of  $A$  with weights the components of  $\mathbf{x}$ .

We can create a function  $T$  from  $\mathbb{R}^n$  to  $\mathbb{R}^m$  by  $T(\mathbf{x}) = A\mathbf{x}$ .

## Example of a Linear Transformation

$$T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

$$T(\mathbf{x}) = A\mathbf{x} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1x_1 + 2x_2 + 3x_3 \\ 4x_1 + 5x_2 + 6x_3 \\ 7x_1 + 8x_2 + 9x_3 \end{bmatrix}$$

## Transformations More Generally

*Definitions:* A **transformation**  $T$  from  $\mathbb{R}^n$  to  $\mathbb{R}^m$  is a rule which assigns to each vector  $\mathbf{x}$  in  $\mathbb{R}^n$  some vector  $T(\mathbf{x})$  in  $\mathbb{R}^m$

The set  $\mathbb{R}^n$  is the **domain** of  $T$  and the set  $\mathbb{R}^m$  is the **codomain** of  $T$

The vector  $T(\mathbf{x})$  is the **image** of  $\mathbf{x}$  under the action of  $T$ .

The set of all images of  $T$  is called the **range** of  $T$ .

## Linear Transformations

We call  $T$  a **matrix transformation** if  $T(\mathbf{x}) = A\mathbf{x}$  for some matrix  $A$ .

*Definition:* A transformation  $T$  from  $\mathbb{R}^n$  to  $\mathbb{R}^m$  is called a **linear transformation** if for all  $\mathbf{u}$  and  $\mathbf{v}$  in the domain of  $T$  and all scalars  $c, d$  both

$$T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}),$$

and

$$T(c\mathbf{u}) = cT(\mathbf{u})$$

**Note: Every matrix transformation is a linear transformation  
but Not all transformations are linear**

**Theorem:** If  $T$  is a linear transformation, then

$$T(\mathbf{0}) = \mathbf{0} \text{ and}$$

$$T(c\mathbf{u} + d\mathbf{v}) = cT(\mathbf{u}) + dT(\mathbf{v})$$

**Proof:**

$$T(c\mathbf{u} + d\mathbf{v}) = T(c\mathbf{u}) + T(d\mathbf{v}) = cT(\mathbf{u}) + dT(\mathbf{v})$$

## Example of a Linear Transformation

$$T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$$

$$T(\mathbf{x}) = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1x_1 + 2x_2 + 3x_3 \\ 4x_1 + 5x_2 + 6x_3 \end{bmatrix}$$

## Another Example of a Linear Transformation

$$T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$$

$$T(\mathbf{x}) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1x_1 + 2x_2 \\ 3x_1 + 4x_2 \\ 5x_1 + 6x_2 \end{bmatrix}$$

## Examples of Nonlinear Transformations

$$T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$$

$$T(\mathbf{x}) : \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \rightarrow \begin{bmatrix} x_1 x_2 x_3 \\ 4x_1 + 5x_2 + 6x_3 \end{bmatrix}$$

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$$T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$$

$$T(\mathbf{x}) : \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \rightarrow \begin{bmatrix} x_1^2 + x_2^2 + x_3 \\ 4x_1 - 5x_2 \\ \ln(x_2 x_3) \end{bmatrix}$$

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$$T : \mathbb{R}^1 \rightarrow \mathbb{R}^1 \text{ by } T(x) = \sin x$$

Special Transformations: **shear, dilation, contraction**