Mathematical Modelling, Vol. 3, pp. 221–239, 1982
0270–0255/82/030221–19$03.00/0

LEONTIEF MODELS, MARKOV CHAINS, SUBSTOCHASTIC MATRICES, AND POSITIVE SOLUTIONS OF MATRIX EQUATIONS

BRUCE PETERSON AND MICHAEL OLINICK
Middlebury College
Middlebury, VT 05753

Communicated by Richard Bellman

Abstract—Many applications of linear algebra call for determining solutions of systems of linear equations with certain prescribed properties. The Leontief input–output models in economics, for example, requires a solution vector all of whose components are nonnegative. The location of an equilibrium distribution for a regular Markov process involves finding a vector each of whose entries is positive. This paper presents an elementary constructive algebraic proof for finding such solutions and for discovering conditions under which they exist. The results extend the theory of solutions to such systems to matrix equations over arbitrary, not necessarily complete, ordered fields.

INTRODUCTION

In many applications of linear algebra, one is interested in obtaining solutions of systems of linear equations with certain prescribed properties. For example, in a closed Leontief input–output model (see below) an economist wishes to study solutions of a system $BX = 0$ in which the components of the vector X must be *nonnegative.* If A represents the transition matrix for a regular Markov chain (see Sec. 4), then one seeks solutions of $AX = X$ in which all the components of X are probabilities; that is, numbers between 0 and 1. In many other planning or production models, the only solutions of relevance are those which are integers.

While the standard theorems of elementary linear algebra tell us a great deal about the existence of solutions to systems of equations, they say little about obtaining solutions having such additional restrictions. In many texts dealing with such applications, reference is given to some powerful existence theorems (e.g., Brouwer's Fixed Point Theorem) which guarantee that such special solutions may be found, but no procedure is described for finding them. In this paper, we will present an elementary constructive proof that can be used in many such situations.

The primary motivation for developing the results discussed here lay in attempting to find a complete treatment of two simple input–output Leontief models that can be presented to a beginning linear algebra class. Since the proofs are strictly algebraic, in contrast to the alternate approaches (e.g., through power series with matrix argument) these results extend the theory to matrices over arbitrary, not necessarily complete, ordered fields.

The closed model

Consider an economy made up of n industries $I_1, I_2 \ldots I_n$. In a certain time period, each industry produces an output of some good or service which is completely utilized by itself or other industries in a predetermined manner which remains constant during that time period. To simplify the presentation here, we suppose that units are chosen so that each industry produces exactly one unit of its product in the given time period.

Let a_{ij} be the fraction of the total output of industry I_j used by industry I_i. Then each a_{ij} is nonnegative and

$$a_{ij} + a_{2j} + \cdots + a_{nj} = 1$$

for each j. The $n \times n$ matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & & \cdots & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

is called the *exchange* or *input–output* matrix. For each industry I_j, let $x_j \geq 0$ denote the price of one unit of its output and let

$$X = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

be the associated *price vector.* In the time period of interest, industry I_i has an income of x_i and an expenditure of

$$\sum_{j=1}^{n} a_{ij}x_j.$$

One notion of equilibrium in such an economy is that each industry spend no more than it receives; that is,

$$\sum_{j=1}^{n} a_{ij}x_j \leq x_i$$

for each $i = 1, 2 \ldots n$. Suppose that X is a price vector which yields such an equilibrium. Then,

$$x = \sum_i x_i \geq \sum_i \sum_j a_{ij}x_j = \sum_j \sum_i a_{ij}x_j = \sum_j x_j \sum_i a_{ij} = \sum_j x_j \cdot 1 = x$$

so that

$$\sum_i x_i = \sum_i \sum_j a_{ij}x_j$$

and hence

$$x_i = \sum_j a_{ij}x_j$$

for each i. Thus X is an equilibrium vector if and only if $AX = X$. The main problem for the closed model then becomes the following: Given an exchange matrix A, does there exist a vector X of nonnegative components so that $(I - A)X = 0$? See Sec. 4 for a solution to this problem.

The open model

Consider again an economy with n industries and some external source of demand for some of the output of each industry. Interpret a_{ij} as the dollar value of the output of industry I_i needed to produce one dollar's value of output of industry I_j. Then each a_{ij} is nonnegative and we assume

$$\sum_i a_{ij} \leq 1$$

for each j; that is, each industry is profitable.

Let $p_j \geq 0$ be the number of units to be produced by industry I_j and let

$$P = \begin{bmatrix} p_1 \\ \vdots \\ p_n \end{bmatrix}$$

be the associated *production vector*. Then the vector $P - AP = (I - A)P$ has components which give the excess production of each industry. The *consumer* or *external demand* for output of industry i has dollar value d_i. Denote the demand vector by D where

$$D = \begin{bmatrix} d_1 \\ \vdots \\ d_n \end{bmatrix}.$$

The basic question for the open model is as follows: Given a demand vector D, is there a production vector P which meets that demand; that is, $(I - A)P = D$? This question is answered in Sec. 5.

The models described here are generally associated with the name of Wassily Leontief, the economist who pioneered their application to real world economies. For further discussion of the appropriateness of the assumptions and possible generalizations of these simple models, see Refs. 8 or 9.

1. POSITIVITY OF COFACTORS

Throughout our discussion, A will denote an n by n matrix ($n \geq 2$) with entries a_{ij}. For simplicity we will assume that the entries are real numbers. Most of the proofs, however, are valid for matrices over any ordered field. We will say the matrix A is *nonnegative* ($A \geqq 0$) if $a_{ij} \geqq 0$ for all i and j; A is *positive* ($A > 0$) if $A \geqq 0$ and $a_{ij} > 0$ for some i and j; and A is *strictly positive* ($A \gg 0$) if $a_{ij} > 0$ for all i and j. We will assume throughout that $A \geqq 0$ and has column sums not exceeding 1, that is, that

$$\sum_i a_{ij} \leqq 1$$

for all j. Such a matrix is called *substochastic*. A *stochastic* matrix is a substochastic matrix for which each column sum is 1.

We will be seeking solutions to matrix equations and inequalities in the form of column vectors which are nonnegative, positive or strictly positive (with the corresponding meanings for these terms) depending upon what special combination of conditions is placed on A (e.g., what can be said if $A \gg 0$ and

$$\sum_i a_{ij} < 1$$

for a single j?).

Denote the determinant of A by $|A|$, the submatrix obtained by deleting the ith row and jth column of A by $A(i, j)$, and the corresponding cofactor by A_{ij}. Thus, $A_{ij} = (-1)^{i+j}|A(i, j)|$. We will not distinguish among identity matrices of different orders.

Standard theorems of linear algebra offer equivalents for $|A| \neq 0$: invertibility, independence of the rows, one-to-oneness of the corresponding transformation, and so on. What distinguishes this theory is the search for positivity, which in many cases will amount to showing that for some matrix A, the determinant $|A| > 0$. The primary tool for approaching this problem constructively is the following lemma.

Lemma 1.1. If A is a substochastic matrix, then, for all i and j, the cofactor $(I - A)_{ij} \geqq 0$. Moreover, if $A \gg 0$, then $(I - A)_{ij} > 0$.

Proof. We attack by induction, writing $B = I - A$ for simplicity. For $n = 2$,

$$B = \begin{bmatrix} 1 - a_{11} & -a_{12} \\ -a_{21} & 1 - a_{22} \end{bmatrix};$$

and

$$\begin{aligned} B_{12} &= (-1)^{1+2}(-a_{21}) = a_{21} \geqq 0, \\ B_{21} &= (-1)^{2+1}(-a_{12}) = a_{12} \geqq 0, \\ B_{11} &= (-1)^{1+1}(1 - a_{22}) = 1 - a_{22} \geqq a_{12} \geqq 0, \\ B_{22} &= (-1)^{2+2}(1 - a_{11}) = 1 - a_{11} \geqq a_{21} \geqq 0. \end{aligned}$$

Moreover, if $A \gg 0$, each of the final inequalities on the right is strict.

Assuming now that the result has been established for matrices of order less than n, we let A be n by n and distinguish three cases.

Case 1: $j < i$. In this case we compute B_{ij} by a cofactor expansion on the row

$$-a_{j1} - a_{j2} \cdots - a_{j,j-1} - a_{j,j+1} \cdots - a_{jn}.$$

If $j < i$, this row is the jth row of $B(i, j)$. Therefore,

$$\begin{aligned} B_{ij} = (-1)^{i+j}|B(i, j)| = (-1)^{i+j}\{&-a_{j1}B(i, j)_{j1} - a_{j2}B(i, j)_{j2} - \cdots \\ &- a_{j,j-1}B(i, j)_{j,j-1} - a_{j,j+1}B(i, j)_{jj} \cdots - a_{jn}B(i, j)_{j,n-1}\}. \end{aligned} \tag{1.1}$$

Now

$$\begin{aligned} B(i, j)_{jk} &= (-1)^{j+k}|B(i, j)(j, k)| = (-1)^{j+k}|B(j, j)(i - 1, k)| \\ &= (-1)^{j+k}(-1)^{i-1+k}B(j, j)_{i-1,k} = (-1)^{i+j-1}B(j, j)_{i-1,k}. \end{aligned} \tag{1.2}$$

The critical observation here is that initially deleting the ith row and then the jth row has the same effect as deleting first the jth row and then the $(i-1)$th row. This is expressed in the second equality of Eq. (1.2) and makes the induction step possible.

The induction hypothesis certainly applies to the matrix $A(j,j) = I - B(j,j)$, since deletion of rows and columns can neither decrease any entry or increase any column sum. Hence $B(j,j)_{i-1,k} \geqq 0$ and every term inside the brackets of (1.1) is of the form

$$-a_{jk}B(i,j)_{jk} \quad \text{or} \quad -a_{j,k+1}B(i,j)_{jk}.$$

But every $a_{ij} \geqq 0$, and we have just seen that every $B(i,j)_{jk}$ has sign $(-1)^{i+j-1}$ or is 0. Inside the brackets then, each term has the same sign, and

$$B_{ij} = (-1)^{i+j}\{0\text{'s or terms with sign } (-1)\cdot(-1)^{i+j-1}\} \geqq 0.$$

Moreover, if $A \gg 0$, then $a_{jk} > 0$ and $B(j,j)_{i-1,k} > 0$ by the induction hypothesis, which assures us that none of the terms inside the brackets is 0. Hence, $B_{ij} > 0$.

Case 2: $j > i$. If $j > i$, the row used for expansion of B_{ij} is the $(j-1)$th row of $B(i,j)$ and the proof proceeds by similar steps, observing in this case that

$$B(i,j)(j-1,k) = B(j,j)(i,k).$$

Case 3: $i = j$. To avoid confusion with running subscripts, we will compute B_{kk}, where k is a fixed subscript. The technique is to replace B by the singular matrix C obtained when the kth row of B is replaced by the sum of all the other rows. Since only the kth row is altered, $C_{kj} = B_{kj}$ for all j.

Formally, we let

$$c_{kj} = \sum_{\substack{i \\ i \neq k}} a_{ij}$$

$$C = \begin{bmatrix} 1-a_{11} & -a_{12} & \cdots & -a_{1k} & \cdots & -a_{1n} \\ -a_{21} & 1-a_{22} & \cdots & -a_{2k} & \cdots & -a_{2n} \\ & & \cdots & & & \\ -a_{k-1,1} & -a_{k-1,2} & \cdots & -a_{k-1,k} & \cdots & -a_{k-1,n} \\ 1-c_{k1} & 1-c_{k2} & \cdots & -c_{kk} & \cdots & 1-c_{kn} \\ -a_{k+1,1} & -a_{k+1,2} & \cdots & -a_{k+1,k} & \cdots & -a_{k+1,n} \\ & & \cdots & & & \\ -a_{n1} & -a_{n2} & \cdots & -a_{nk} & \cdots & 1-a_{nn} \end{bmatrix}$$

Computing $|C|$ by a cofactor expansion on the kth row we have

$$0 = |C| = (1-c_{k1})C_{k1} + (1-c_{k2})C_{k2} + \cdots + (-c_{kk})C_{kk} + \cdots + (1-c_{kn})C_{kn}. \tag{1.3}$$

Now $c_{kj} \leqq 1$ for every j, and we know from case 1 that $C_{kj} = B_{kj} \geqq 0$ for $j \neq k$. Therefore, the right hand side of (1.3) consists of nonnegative terms and $-c_{kk}C_{kk}$. It follows at once that $c_{kk}B_{kk} = c_{kk}C_{kk} \geqq 0$. If $c_{kk} > 0$, then certainly

$$B_{kk} \geqq 0.$$

If $c_{kk} = 0$, then the kth column of A is all 0 except possibly for a_{kk}. (A mathematical possibility, but not a very realistic economic situation.) In this case, consider the matrix A^* obtained from A by replacing the kth column of A by a vector all of whose entries are $1/n$.

Everything established so far applies to A^*, but $c^*_{kk} = (n-1)/n > 0$. Hence,

$$B_{kk} = B^*_{kk} \geqq 0.$$

If $A \gg 0$, then, from the previous case, $B_{kj} > 0$ for $k \neq j$, and

$$0 < \sum_i a_{ij} = c_{kj} + a_{kj} \leqq 1,$$

which implies that

$$0 < a_{kj} \leqq 1 - c_{kj}.$$

In this situation then, the right hand side of (1.3) consists of positive terms and $-c_{kk}C_{kk}$. Since $c_{kk} > 0$, necessarily $C_{kk} > 0$. This completes the proof.

If may be of interest to note that the hypotheses of the second part of this lemma may be weakened slightly. If we assume only that $a_{ij} > 0$ for $i \neq j$ and $a_{ii} < 1$ for all i (i.e., allow the possibility that $a_{ii} = 0$), the proof that $(I-A)_{ij} > 0$ for all i and j is unaffected. This is of some importance economically since an industry may well consume none of its own output.

2. INVERTIBILITY OF $I - A$

With the help of Lemma 1.1, we can now learn a great deal about the invertibility of the matrix $I - A$ and the sign of its determinant.

Corollary 2.1. If A is a substochastic matrix, then $|I - A| \geqq 0$.

Proof. Applying the lemma to the $(n+1) \times (n+1)$ substochastic matrix A^*, where

$$A^* = \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & A \end{array}\right], \qquad |I - A| = (I - A^*)_{11} \geqq 0.$$

Corollary 2.2. If A is a substochastic matrix and $I - A$ is invertible, then $(I-A)^{-1} > 0$. Moreover, if $A \gg 0$, then $(I-A)^{-1} \gg 0$.

Proof. From Corollary 2.1 and the hypothesis, $|I - A| > 0$. Since

$$(I-A)^{-1} = \frac{\operatorname{adj}(I-A)}{|I-A|}$$

where $\operatorname{adj}(I-A)$, the classical adjoint matrix, is the transpose of the matrix of cofactors, $(I-A)^{-1} \geqq 0$. The inverse cannot be the 0 matrix, so necessarily $(I-A)^{-1} > 0$.

The second part of Lemma 1 guarantees that every cofactor of $I - A$, for a strictly positive A, is positive, and therefore that $(I-A)^{-1} \gg 0$.

Corollary 2.3. If A is a strictly positive substochastic matrix, then $\operatorname{rank}(I - A) \geqq n - 1$.

Proof. Every $(n-1)$ by $(n-1)$ submatrix of $I - A$ has nonzero determinant.

A more general setting is useful in establishing conditions for the invertibility of $I - A$. We will define for any matrix D a norm

$$\|D\| = \max_j \sum_i |d_{ij}|.$$

If X is a column vector, then

$$\|X\| = \sum_i |x_i|.$$

For any column vector X,

$$\|DX\| = \sum_i \left|\sum_k d_{ik}x_k\right| \leqq \sum_i \sum_k |d_{ik}| \cdot |x_k| = \sum_k \sum_i |d_{ik}| \cdot |x_k| = \sum_k |x_k| \sum_i |d_{ik}|$$
$$\leqq \|D\| \sum_k |x_k| = \|D\| \cdot \|X\|.$$

That is,

$$\|DX\| \leqq \|D\| \cdot \|X\|.$$

If $I - D$ is singular, then there is some nonzero column vector X such that $DX = X$. Since $\|X\| = \|DX\| \leqq \|D\| \cdot \|X\|$, if $\|D\| < 1$, we are led to the absurdity $\|X\| < \|X\|$. We have proved the following lemma.

Lemma 2.4. If $\|D\| < 1$, then $I - D$ is nonsingular.

Returning to the original setting and strengthening slightly the conditions on A we have immediately

Corollary 2.5. If $A \geqq 0$ and $\Sigma_i a_{ij} < 1$ for every j, then $|I - A| > 0$.

Corollary 2.6. If $A \gg 0$, no column sum exceeds 1, and $\Sigma_i a_{ij} < 1$ for some j, then $|I - A| > 0$.

Proof. The idea here is to apply Corollary 2.5 to the matrix A^2. Clearly $A^2 \gg 0$, and if the elements of A^2 are b_{ij}'s, then for every j,

$$\sum_i b_{ij} = \sum_i \left(\sum_k a_{ik}a_{kj}\right) = \sum_k \sum_i a_{ik}a_{kj} = \sum_k a_{kj} \sum_i a_{ik} < \sum_k a_{kj} \leqq 1.$$

Therefore,

$$|I - A| \cdot |I + A| = |I - A^2| > 0,$$

and, with the help of Corollary 2.1 again, $|I - A| > 0$.

To recapitulate here, we now know that:

1. If A is nonnegative and *every* column sum is less than 1, then $(I-A)^{-1}$ is positive.
2. If A is strictly positive, no column sum exceeds 1, and *some* column sum is less than 1, then $(I-A)^{-1}$ is strictly positive.
3. For a strictly positive matrix A with all column sums not exceeding 1, $I-A$ is singular if and only if every column sum of A is exactly 1.

Not much improvement on these results seems possible. The positive matrices

$$\begin{bmatrix} 0 & 0 \\ \frac{1}{2} & 1 \end{bmatrix} \qquad \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{4} \\ 0 & 0 & \frac{1}{4} \end{bmatrix}$$

have single columns with sums less than 1, but $|I-A|=0$ in each case.

3. SECTORS OF A SQUARE MATRIX

A *sector* of a square matrix is a submatrix whose diagonal elements are diagonal elements of the original matrix. A *k-sector* is a k by k submatrix obtained this way. If D is an n by n matrix with rank $D=n-1$, then some $n-1$ by $n-1$ submatrix $D(i,j)$ has nonzero determinant, but we cannot be certain that there is an $(n-1)$-sector $D(i,i)$ with $|D(i,i)|\neq 0$. Indeed, each of the upper triangular matrices

$$\begin{bmatrix} 0 & 1 & 1 & 1 & . & . & . & 1 \\ 0 & 0 & 1 & 1 & . & . & . & 1 \\ 0 & 0 & 0 & 1 & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & 0 & 1 \\ 0 & 0 & 0 & 0 & . & . & 0 & 0 \end{bmatrix}$$

has rank$(n-1)$ and no nonsingular sector of any order.

The situation for $I-A$ is somewhat better. We will show that rank$(I-A)=k$ guarantees the existence of a nonsingular k sector. This result in turn is the key to dealing with the closed model for the economy.

Each sector of the matrix A describes the economic intercourse among the industries involved. We will see in this section that the entire economy can be broken down in a natural way into what will later be called *productive* and *nonproductive* sectors corresponding precisely to certain nonsingular and singular sectors of the matrix $I-A$.

Lemma 3.1. If A is a substochastic matrix and $|I-A|\neq 0$, then $|(I-A)(i,i)| = |I-A(i,i)|>0$ for every i.

Proof. If $a_{ii}=1$, the ith column of $I-A$ consists of zeros alone and $|I-A|=0$. Therefore, we can assume that $a_{ii}<1$ for all i. Using a cofactor expansion on the ith row of $I-A$, we have

$$|I-A| = -a_{i1}(I-A)_{i1} - a_{i2}(I-A)_{i2} - \cdots + (1-a_{ii})(I-A)_{ii} - \cdots - a_{in}(I-A)_{in}$$

The left hand side of this equation is positive and for each term on the right with $k\neq i$,

$$-a_{ik}(I-A)_{ik} \leqq 0.$$

Necessarily then,

$$(1-a_{ii})(I-A)_{ii}>0 \quad \text{and} \quad |(I-A)(i,i)|=(I-A)_{ii}>0.$$

The result is not true in more general settings. Any n by n permutation matrix with 0 diagonal is nonsingular with singular $(n-1)$ sectors. The same is true of the strictly positive matrix

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 1 \\ 1 & 2 & 2 \end{bmatrix}$$

The lemma clearly implies that if A is substochastic and $I-A$ is nonsingular, then every sector of $I-A$ is nonsingular, or, equivalently, that the singularity of any sector forces the singularity of all sectors containing it.

It is quite possible for a singular matrix to have nonsingular sectors, and indeed just such matrices are characteristic of the economic situation. They take a particularly nice form.

Lemma 3.2. If A is a substochastic matrix, and if $|I-A|=0$ while $|I-A(k,k)|\neq 0$, then $\Sigma_i a_{ik}=1$.

Proof. Let $s_j=\Sigma_i a_{ij}$ and consider the matrix

$$A^*=\begin{bmatrix} 1-a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & 1-a_{22} & \cdots & -a_{2n} \\ & & \cdots & \\ -a_{k-1,1} & -a_{k-1,2} & \cdots & -a_{k-1,n} \\ 1-s_1 & 1-s_2 & \cdots & 1-s_n \\ -a_{k+1,1} & -a_{k+1,2} & \cdots & -a_{k+1,n} \\ & & \cdots & \end{bmatrix},$$

which results from adding all the other rows to the kth row. Since the only change has been in the kth row, $A^*_{kj}=(I-A)_{kj}\geqq 0$ for every j. Moreover,

$$0=|I-A|=|A^*|=\sum_j (1-s_j)A^*_{kj}.$$

All the terms on the right are nonnegative, and $A^*_{kk}=(I-A)_{kk}>0$ by hypothesis. We must, therefore, have $s_k=1$.

Corollary 3.3. If A is a substochastic matrix, and if $|I-A|=0$, while $|I-A(j,j)|\neq 0$ for all j, then all the columns of A sum to 1.

The *initial k-sector* of an $n\times n$ matrix is the k-sector obtained by deleting the last $n-k$ rows and columns.

Corollary 3.4. Let A be a substochastic matrix. If B is a singular initial k-sector of $I-A$, and if every $(k-1)$-sector of B is nonsingular, then $I-A$ has the special form

$$I-A=\left[\begin{array}{c|c} B & C \\ \hline 0 & D \end{array}\right].$$

Proof. The columns of B sum to 1. Hence, in $I-A$, the elements of these columns which are not in B must all be 0.

The partitioning of $I - A$ is now easy. By interchanging rows and columns we may assume that

$$a_{ii} \begin{cases} <1 \text{ for } i = 1, 2 \ldots m \\ =1 \text{ for } i = m+1, m+2 \ldots n. \end{cases}$$

so that the first m columns of $I - A$ are nonzero and the last $n - m$ are identically 0. We search in the first m columns for a singular 2-sector. Either we find one or all the 2-sectors are nonsingular. In the latter case, we search for a singular 3-sector. Continuing in this fashion we find either

(1) that the first m rows and columns form a nonsingular sector or

(2) a sector B_1 such that $|B_1| = 0$ and $|B_1(i, i)| \neq 0$ for every i.

By Corollary 3.4, $I - A$ has (up to permutations of subscripts) the form

$$I - A = \left[\begin{array}{c|c|c} B_1 & C_1 & \\ \hline 0 & B_1' & 0 \\ \hline 0 & D_1 & \end{array}\right].$$

Now we concentrate on the sector B_1', and, just as before, find either that $|B_1'| \neq 0$ or that B_1' contains a singular sector B_2 all of whose subsectors are nonsingular.

The process continues until some remnant sector B_t' is nonsingular. Setting $B_t' = P$, for reasons to become clear later, the matrix $I - A$, after appropriate rearranging of course, takes the particularly nice block form:

$$I - A = \left[\begin{array}{cccc|c|c} B_1 & & & & & \\ & B_2 & & \mathbf{0} & & \\ & & \ddots & & C & \mathbf{0} \\ & & & B_t & & \\ \hline & \mathbf{0} & & & P & \\ \hline & & & & D & \end{array}\right],$$

where each B_i is a singular sector each of whose proper subsectors is nonsingular, and P is a nonsingular sector.

We know already that any sector containing a singular sector is necessarily singular itself. Note now that, if B is any sector without 0 columns, it has the block form

$$B = \left[\begin{array}{cccc|c} B_1^* & & & & \\ & B_2^* & & \mathbf{0} & \\ & & \ddots & & C^* \\ & \mathbf{0} & & B_t^* & \\ \hline & & & & P^* \end{array}\right],$$

where B_i^* is a subsector of B_i, P^* a subsector of P, and C^* the appropriate submatrix of C. Now

$$|B| = \Big(\prod_i |B_i^*|\Big)\,|P^*|.$$

Therefore B is singular if and only if $B_i^* = B_i$ for some i. In other words, the decomposition of $I - A$ is unique in the sense that every singular sector must contain one of the B_i. We have proved the following theorem.

THEOREM 3.5. If A is a substochastic matrix, then, after an appropriate renumbering of rows and columns,

$$I - A = \begin{bmatrix} B_1 & & & \mathbf{0} & & & \\ & B_2 & & & & & \\ & & \ddots & & & C & \mathbf{0} \\ & \mathbf{0} & & & B_t & & \\ & & & & & P & \\ & & & & & D & \end{bmatrix}$$

where each B_i is a singular sector each of whose proper subsectors is nonsingular and P is a nonsingular sector. The decomposition is unique in the sense that any singular sector without zero columns must contain one of the B_i's. For each sector B_i, we have

$$\operatorname{rank}(B_i) = k_i = n_i - 1 = o(B_i) - 1,$$
$$\operatorname{rank}(P) = k_{t+1} = o(P),$$

and

$$\operatorname{rank}(I - A) = \sum_{i=1}^{t+1} k_1.$$

Now delete from $I - A$ the last $n - m$ rows and columns, and the first row and column of each sector B_i, and all the other elements of $I - A$ in the deleted rows and columns to form a matrix $\hat{B}$,

$$\hat{B} = \begin{bmatrix} \hat{B}_1 & & & & \\ & \hat{B}_2 & & & \\ & & \ddots & & \hat{C} \\ & & & \hat{B}_t & \\ & & & & P \end{bmatrix},$$

where $\hat{B}_i$ is a k_i-subsector of B_i. Then $|\hat{B}_i| \neq 0$ for all i and

$$|\hat{B}| = \left(\prod_{i=1}^{t} |\hat{B}_i|\right) \cdot |P| \neq 0.$$

Moreover,

$$o(\hat{B}) = \sum_{i=1}^{t} o(\hat{B}_i) + o(P) = \sum_{i=1}^{t} (n_i - 1) + k_{t+1} = \sum_{i=1}^{t+1} k_i = \text{rank}(I - A).$$

We have proved the following theorem.

THEOREM 3.6. If A is a substochastic matrix and $\text{rank}(I - A) = k$, then $I - A$ contains a nonsingular k-sector.

4. THE CLOSED MODEL AND MARKOV CHAINS

We deal here with stochastic matrices, that is nonnegative matrices for which each column sums to 1. Economically, the system can be interpreted either as one in which there is no demand or one in which demand is considered as an industry which consumes all of its own output.

Such matrices also arise in the analysis of finite Markov chains: stochastic processes in which the probability of being in a particular state at any step depends only on the state occupied at the previous step. More exactly, consider repetitive trials of an experiment with a finite number of possible outcomes $S_1, S_2 \ldots S_n$. The sequence of outcomes is a *Markov chain* if there is a set of n^2 numbers $\{p_{ij}\}$ such that the conditional probability of outcome S_j on any trial, given outcome S_i on the previous trial, is p_{ij}; that is,

$$p_{ij} = \Pr(S_j \text{ on trial } k+1 | S_i \text{ on trial } k), \quad 1 \leq i, j \leq n, \ k = 1, 2 \ldots$$

The transition probabilities p_{ij} can be arranged in a stochastic matrix

$$p = \begin{bmatrix} p_{11} & p_{21} & \cdots & p_{n1} \\ p_{12} & p_{22} & \cdots & p_{n2} \\ & & \cdots & \\ p_{1n} & p_{2n} & \cdots & p_{nn} \end{bmatrix}.$$

Let $p_i^{(k)}$ denote the probability that the outcome on the kth trial is S_i and let $\mathbf{p}^{(k)} = (p_1^{(k)}, p_2^{(k)} \ldots p_n^{(k)})^T$ be the associated probability distribution vector. Then, for a Markov process, we have

$$\mathbf{p}^{(k+1)} = P\mathbf{p}^{(k)}, \quad k = 1, 2 \ldots$$

Thus a Markov chain is completely characterized by its transition matrix P and an initial probability distribution $\mathbf{p}^{(0)}$ in the sense that $\mathbf{p}^{(k)} = P^k \mathbf{p}^{(0)}$, $k = 0, 1, 2 \ldots$

In many applications of Markov processes, one is interested in the existence of equilibrium probability distributions; that is, vectors $\mathbf{p}$ such that $\mathbf{p} = P\mathbf{p}$. For example, if some positive power of P is strictly positive then it can be shown that there is a unique strictly positive distribution p so that

$$\lim_{k \to \infty} P^k \mathbf{p}^{(0)} = \mathbf{p}$$

for any initial distribution $\mathbf{p}^{(0)}$. For further discussion, applications and proofs see Refs. 1, 3, 4, 7, and 10.

The existence of equilibrium distribution vectors can be proven using the heavy artillery of the Brouwer Fixed Point Theorem. If S is the set of all probability distributions under consideration, then S forms the standard $(n-1)$-simplex in R^n and for p in S, we have

$$\sum_i (P\mathbf{p})_i = \sum_i \sum_j (p_{ji}p_j) = \sum_j \sum_i p_{ji}p_j = \sum_j p_j \sum_i p_{ji} = \sum_j p_j = 1,$$

so that $P\mathbf{p}$ also lies in S. Thus, we can view P as a linear transformation from S to itself. Since P is continuous, Brouwer's Theorem guarantees the existence of an equilibrium vector.

We can consider our exchange matrix A as the transition matrix of a Markov chain provided we consider only *normalized* price vectors X whose components sum to 1. It is the purpose of this section to show a constructive proof for existence of equilibrium vectors.

The problem here is to find a positive solution X to the matrix equation $AX = X$; that is, a solution X in which all components are nonnegative and at least one is positive. Since each column of A sums to 1, the matrix $I - A$ is row equivalent to one with a 0 row. Hence, $\text{rank}(I-A) \leqq n-1$, and there is a nontrivial solution of $(I-A)X = 0$. To show that the solution can actually be taken positive, we prove the more general result:

THEOREM 4.1. If A is a substochastic matrix, $\text{rank}(I-A) = k < n$, then the solution space of the equation $(I-A)X = 0$ has a basis consisting of positive vectors.

The proof of this theorem requires only familiar linear algebra and the results on cofactors and sectors already developed. The desired solution vectors can be computed directly from the matrix $I - A$. Recall first that if C is an n by n matrix and

$$W_j = \begin{bmatrix} C_{j1} \\ C_{j2} \\ \vdots \\ C_{jn} \end{bmatrix},$$

then

$$CW_j = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ |C| \\ \vdots \\ 0 \end{bmatrix}$$

where the $|C|$ occurs in the jth component.

Trivially then, if $\text{rank}(C) < n$, certainly $CW_i = 0$.

To prove the theorem we take $C = I - A$. From Lemma 1.1, we know that $W_j \geqq 0$. We need only to show that judicious choice of j can assure $W_j \neq 0$.

Proof of the theorem. The low rank cases are instructive. If $\text{rank}(I-A) = 0$, then $I - A = 0$ and any positive vector is a solution to $(I-A)X = 0$. The standard basis for R^n is a positive basis for the solution space.

If $1 - a_{ii} = 0$, then $a_{ji} = 0$ for $j \neq i$, and the ith column of $I - A$ is identically 0. Therefore, $\text{rank}(I-A) = k$ assures that $a_{ii} < 1$ for at least k different i's. If $\text{rank}(I-A) = 1$, then there is some subscript j, so that $a_{jj} < 1$.

In this case, since every row of $I-A$ is a multiple of the jth row, it is sufficient to find positive solutions to the equation

$$[-a_{j1} \quad -a_{j2} \quad \cdots \quad 1-a_{jj} \quad \cdots \quad -a_{jn}]\cdot X = 0.$$

The vectors

$$Z_1=\begin{bmatrix}1-a_{jj}\\0\\0\\\vdots\\0\\a_{j1}\\0\\\vdots\\0\end{bmatrix},\quad Z_2=\begin{bmatrix}0\\1-a_{jj}\\0\\\vdots\\0\\a_{j2}\\0\\\vdots\\0\end{bmatrix}\ \ldots\ Z_{j-1}=\begin{bmatrix}0\\0\\\vdots\\1-a_{jj}\\a_{j,j-1}\\0\\\vdots\\0\end{bmatrix}\ Z_{j+1}=\begin{bmatrix}0\\0\\\vdots\\0\\a_{j,j+1}\\1-a_{jj}\\0\\\vdots\\0\end{bmatrix}\ \ldots\ Z_n=\begin{bmatrix}0\\0\\\vdots\\0\\a_{jn}\\0\\\vdots\\0\\1-a_{jj}\end{bmatrix}$$

are all positive since $1-a_{jj}>0$; they are independent by inspection. Since the dimension of the solution space is $n-1$,

$$\{Z_1, Z_2 \ldots Z_{j-1}, Z_{j+1} \ldots Z_n\}$$

is the desired positive basis.

In general, if $\operatorname{rank}(I-A)=k<n$, there is a k-sector B in $I-A$ with $|B|\neq 0$. Assuming that B uses the first k rows and columns of $I-A$, every other row is necessarily a linear combination of the first k and it is sufficient to solve the system:

$$\left[\begin{array}{c|cccc} & -a_{1,k+1} & -a_{1,k+2} & \cdots & -a_{1,n}\\ B & -a_{2,k+1} & -a_{2,k+2} & \cdots & -a_{2,n}\\ & & & \vdots & \\ & -a_{k,k+1} & -a_{k,k+2} & \cdots & -a_{k,n}\end{array}\right]\cdot X = 0. \tag{4.1}$$

We first consider the $n-k$ matrices,

$$I-A_j=\begin{bmatrix} & & & & -a_{1j}\\ & & B & & -a_{2j}\\ & & & & \vdots\\ & & & & -a_{kj}\\ -a_{j1} & -a_{j2} & \cdots & -a_{jk} & 1-a_{jj}\end{bmatrix},$$

for $k+1\leqq j\leqq n$. Each A_j is substochastic, so that all the cofactors of $I-A_j$ are nonnegative. Moreover,

$$(I-A_j)_{k+1,k+1}=|B|>0.$$

It follows at once that

$$Y_j=\begin{bmatrix}(I-A_j)_{k+1,1}\\(I-A_j)_{k+1,2}\\ \vdots\\(I-A_j)_{k+1,k+1}\end{bmatrix}$$

is a positive solution for $(I - A_j)X = 0$, and

$$Z_j = \begin{bmatrix} (I-A_j)_{k+1,1} \\ (I-A_j)_{k+1,2} \\ \vdots \\ (I-A_j)_{k+1,k} \\ 0 \\ 0 \\ \vdots \\ (I-A_j)_{k+1,k+1} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{matrix} \\ \\ \\ \\ \\ \leftarrow j\text{th component} \\ \\ \\ \\ \\ \\ \end{matrix}$$

is a positive solution to (4.1). The dimension of the solution space of $I - A$ is $n - k$, and the vectors Z_j are obviously independent. Hence, the desired basis of positive solutions to $(I - A)X = 0$ is $\{Z_{k+1}, Z_{k+2} \dots Z_n\}$.

Corollary 4.2. If $A \gg 0$ and stochastic, then there is a solution $X \gg 0$ to $(I - A)X = 0$, which is unique up to scalar multiples.

Proof. Since all the columns sum to 1, $\operatorname{rank}(I - A) \leqq n - 1$. Since $A \gg 0$, by Corollary 2.3, $\operatorname{rank}(I - A) \geqq n - 1$. If B is a nonsingular $(n - 1)$-sector of $I - A$, then

$$I - A = \left[\begin{array}{ccc|c} & \boldsymbol{B} & & \begin{matrix} -a_{1n} \\ -a_{2n} \\ \vdots \end{matrix} \\ \hline -a_{n1} & \cdots & & 1 - a_{nn} \end{array}\right]$$

and

$$X = \begin{bmatrix} (I-A)_{n1} \\ (I-A)_{n2} \\ \vdots \\ (I-A)_{nn} \end{bmatrix}$$

is a solution to $(I - A)X = 0$. From Lemma 1.1, $X \gg 0$. The uniqueness follows from the fact that the dimension of the solution space is $n - (n - 1) = 1$.

Corollary 4.3. If A is a stochastic matrix and $A^m \gg 0$ for some positive integer m, then there is a solution $X \gg 0$ to $(I - A)X = 0$, which is unique up to scalar multiples.

Proof. By a standard argument every positive power of a stochastic matrix is stochastic. If X is a strictly positive solution to the equation $(I - A^m)X = 0$, then

$$(I - A^m)AX = (A - A^{m+1})X = A(I - A^m)X = 0,$$

and AX is also a solution. From Corollary 4.2, $AX = \alpha X$ for some scalar. Since A is positive and X is strictly positive, $\alpha > 0$. Moreover, $X = A^n X = \alpha^n X$ implies that $\alpha^n = 1$. Because $\alpha = -1$ is absurd, $\alpha = 1$ and $AX = X$.

If $Y \gg 0$ is any solution to $(I - A)Y = 0$, then $(I - A^n)Y = 0$ and Y is a scalar multiple of X from Corollary 4.2.

5. THE OPEN MODEL

In the open model for an economy some output is accounted for by consumer demand. Every closed model may be considered as an open model also. The industries in this system may then account for less than 100% of the output of a particular industry. In terms of our matrix A, this means that some columns may sum to less than 1. We hope for a system in which output exceeds input, so that demand can be met. We will say the system, or the matrix A, is *productive* if there is a nonnegative vector X (output) such that $X \gg AX$. The vector X is a *production vector.* Productivity of A has a particularly nice equivalent.

THEOREM 5.1. A substochastic matrix A is productive if and only if $I - A$ is nonsingular.

Proof. We may as well assume then that $A > 0$ since the theorem is trivial otherwise. Let A be productive, and let $X \geqq 0$ be a production vector guaranteed by the definition. Then,

$$X \gg AX \geqq 0.$$

Now $AX \ll X$ implies that

$$\sum_j a_{ij}x_j < x_i \quad \text{for} \quad i = 1, 2 \ldots n \tag{5.1}$$

and

$$\sum_j \Big(\sum_i a_{ij}\Big)x_j = \sum_i \sum_j a_{ij}x_j < \sum_i x_i;$$

Therefore, for some k,

$$\sum_i a_{ik} < 1. \tag{5.2}$$

Inequality (5.1) also implies that

$$0 \leqq \sum_{j \neq i} a_{ij}x_j < x_i - a_{ii}x_i = (1 - a_{ii})x_i. \tag{5.3}$$

Therefore $a_{ii} < 1$ for every i. The proof is by induction on the order of A. When $n = 1$, we have

$$|I - A| = 1 - a_{11} > 0$$

and $I - A$ is nonsingular. The case for $n = 2$ is also instructive,

$$I - A = \begin{bmatrix} 1 - a_{11} & -a_{12} \\ -a_{21} & 1 - a_{22} \end{bmatrix}.$$

From the column sum condition,

$$a_{21} \leqq 1 - a_{11} \quad \text{and} \quad a_{12} \leqq 1 - a_{22},$$

and from (5.2), at least one of these inequalities is strict. Therefore,

$$|I - A| = (1 - a_{11})(1 - a_{22}) - a_{12}a_{21} > 0$$

and $I - A$ is nonsingular.

Now assume the theorem proved for matrices of order $n - 1$ and let A be an n by n productive matrix with associated nonnegative production vector X. We will concentrate our attention on the sector $A(i, i)$, which is certainly substochastic. Now $(I - A)X \gg 0$ implies that

$$[I - A(i, i)]\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{i-1} \\ x_{j-1} \\ \vdots \\ x_n \end{bmatrix} \gg \begin{bmatrix} a_{1i}x_i \\ a_{2i}x_i \\ \vdots \\ a_{i-1,i}x_i \\ a_{i+1,i}x_i \\ \vdots \\ a_{ni}x_i \end{bmatrix} \geqq 0.$$

Therefore $A(i, i)$ is productive and $I - A(i, i)$ is nonsingular.

Now let

$$s_j = \sum_i a_{ij}.$$

We know from hypothesis that $s_j \leqq 1$ for all j and from (5.2) that $s_k < 1$ for some k. By Corollary 3.2 then, $|I - A| \neq 0$.

To prove the converse, note first that for any square matrix B, if

$$x_j = \sum_i B_{ij} \quad \text{and} \quad X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix},$$

then $X = W_1 + W_2 + \cdots + W_n$ as in Theorem 4.1. Therefore,

$$BX = B(W_1 + W_2 + \cdots + W_n) = \begin{bmatrix} |B| \\ |B| \\ \vdots \\ |B| \end{bmatrix}$$

In the setting of our theorem, we may take $B = I - A$. Hence $|B| > 0$ and $(I - A)X \gg 0$. This completes the proof.

Productivity guarantees that demand can be met. In the corollary we think of D as a demand vector and X as the production vector required to meet that demand.

Corollary 5.2. If A is a productive substochastic matrix, then the equation $(I - A)X = D$ has

(i) a nonnegative solution if $D \geq 0$;
(ii) a positive solution if $D > 0$;
(iii) a strictly positive solution if $D \gg 0$.

Proof. From Theorem 5.1, $I - A$ is invertible and necessarily then $(I - A)^{-1} > 0$. The solution vector is $X = (I - A)^{-1}B$. Only part (iii) needs comment, and this follows at once since every row of $(I - A)^{-1}$ must contain a positive entry.

A fundamental problem in any ecomonic system is to meet demand. Our theory tells us conditions under which demand can be met and provides a simple construction for the production vector required to meet any particular demand. At least the construction is simple in theory. In practice, however, it involves computations of determinants and inverses of very large matrices.

The task is simplified somewhat by the decomposition of the matrix $I - A$ into singular and nonsingular sectors. In our new terminology, Theorem 3.5 describes a decomposition of the economy into (1) a number of nonproductive sectors, $B_1, B_2 \ldots B_t$, which are minimal in the sense that all their proper subsectors are productive, and (2) a productive sector P. Moreover, this decomposition is unique in the sense that the only nonproductive sectors are the B_i's and sectors containing them or, equivalently, that a sector is productive if and only if it contains no B_i.

Of course some of the nonproductive sector may be rather trivial. A nonproductive sector B_i consisting of a single industry I_k has $|B_i| = 1 - a_{kk} = 0$. Therefore, $a_{kk} = 1$ and $a_{ik} = 0$ for $i \neq k$. In other words, a single industry is nonproductive if and only if it consumes all its own output. Demand can be considered as a nonproductive industry.

Eliminating all nonproductive industries from consideration, the remaining nonproductive sectors are nontrivial. If B_i is such a sector, then eliminating from it one industry (any one!) results in a productive subsector B_i' with one fewer industries. If this is done in each nonproductive sector the resulting economy has a matrix

$$I - B = \left[\begin{array}{cccc|c} B_1' & & & 0 & \\ & B_2' & & & \\ & & \ddots & & C \\ 0 & & & B_t' & \\ \hline & & & & P \end{array}\right],$$

which is productive since

$$|I - B| = \left(\prod_i |B_i'|\right) \cdot |P| > 0.$$

The number t, of minimal nonproductive sectors, is the minimum number of industries which must be allowed to fail in order to transform a nonproductive economy into a productive one.

Determining $(I - B)^{-1}$ is noticeably simplified by this decomposition, since the inverses of the B_i' and of P, if the sectors are not too large, are comparatively easy to compute. In particular if

$$Q = \begin{bmatrix} B_1' & & & 0 \\ & B_2' & & \\ & & \ddots & \\ 0 & & & B_t' \end{bmatrix}$$

then, of course,

$$Q^{-1} = \begin{bmatrix} (B_1')^{-1} & & & \mathbf{0} \\ & (B_2')^{-1} & & \\ & & \ddots & \\ \mathbf{0} & & & (B_t')^{-1} \end{bmatrix}$$

and

$$I - B = \left[\begin{array}{c|c} Q & C \\ \hline 0 & P \end{array}\right].$$

It is an easy matter to verify that

$$(I - B)^{-1} = \left[\begin{array}{c|c} Q^{-1} & -Q^{-1}CP^{-1} \\ \hline 0 & P^{-1} \end{array}\right].$$

It is sufficient to compute the inverses of these productive sectors to find $(I - B)^{-1}$.

REFERENCES

1. A. T. Barucha-Reid, *Elements of the Theory of Markov Processes and Their Applications*, McGraw-Hill, New York (1960).
2. C. G. Boot, *Mathematical Reasoning in Economics and the Management Sciences*, Prentice-Hall, Englewood Cliffs, NJ (1967).
3. K. L. Chung, *Elementary Probability Theory with Stochastic Processes*, Springer, New York (1974).
4. K. L. Chung, *Markov Chains with Stationary Transition Probabilities*, Springer, New York (1967).
5. D. P. Gaver and G. L. Thompson, *Programming and Probability Models in Operations Research*, Brooks-Cole, Monterey (1973).
6. S. Karlin, *Mathematical Models and Theory in Games, Programming and Economics*, Vol. 1, Addison-Wesley, Reading, PA (1959).
7. J. G. Kemeny and J. L. Snell, *Finite Markov Chains*, Van Nostrand, New York (1960).
8. W. W. Leontief, *The Structure of the American Economy* 1919–1929, Harvard University Press, Cambridge, MA (1941).
9. W. W. Leontief, *Input–Output Economics*, Oxford University Press, New York (1966).
10. M. Olinick, *An Introduction to Mathematical Models in the Social and Life Sciences*, Addison-Wesley, Reading, MA (1978).