MATH 200C Linear Algebra

Highlights from §1.8: Linear Transformations

Think of $A\mathbf{x} = \mathbf{b}$ as "Matrix A acting on vector $\mathbf{x}$ to produce a vector $\mathbf{b}$"

Or "Multiply A by $\mathbf{x}$ to get a new product $\mathbf{b}$"

If A is $m \times n$ and $\mathbf{x}$ is $n \times 1$ then $\mathbf{b} = A\mathbf{x}$ is $m \times 1$.

Recall: The vector $\mathbf{b} = A\mathbf{x}$ is a linear combination of the columns of A with weights the components of $\mathbf{x}$.

We can create a function T from $\mathbb{R}^n$ to $\mathbb{R}^m$ by $T(\mathbf{x}) = A\mathbf{x}$.

Definitions: A **transformation** T from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a rule which assigns to each vector $\mathbf{x}$ in $\mathbb{R}^n$ some vector $T(\mathbf{x})$ in $\mathbb{R}^m$. The set $\mathbb{R}^n$ is the **domain** of T and the set $\mathbb{R}^m$ is the **codomain** of T. The vector $T(\mathbf{x})$ is the **image** of $\mathbf{x}$ under the action of T. The set of all images of T is called the **range** of T.

Some Examples with $n = 2$.

If $T(\mathbf{x}) = A\mathbf{x}$ for some matrix A, then T is called a **matrix transformation.**

Definition: A transformation T from $\mathbb{R}^n$ to $\mathbb{R}^m$ is called a linear transformation if for all $\mathbf{u}$ and $\mathbf{v}$ in the domain of T and all scalars c, both

(i) $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$, and
(ii) $T(c\,\mathbf{u}) = c\,T(\mathbf{u})$

Note: **Every matrix transformation is a linear transformation.**

Not all transformations are linear

Theorem: If T is a linear transformation, then

$T(0) = 0$ and

$T(c\mathbf{u} + d\mathbf{v}) = c\,T(\mathbf{u}) + d\,T(\mathbf{v})$

Special Transformations: **shear, dilation, contraction**