MATH 200C   Linear Algebra

Highlights from §1.4: The Matrix Equation Ax = b

Definition: If A is an $m \times n$ matrix, with columns $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ and if $\mathbf{x}$ is in $\mathbb{R}^n$, then the product of A and $\mathbf{x}$, denoted by $A\mathbf{x}$, **is the linear combination of the columns of *A* using the corresponding entries in x as weights;** that is,

$$A\mathbf{x} = [\boldsymbol{a_1}\ \boldsymbol{a_2}\ \ldots\ \boldsymbol{a_n}]\begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_n \end{bmatrix} = x_1\boldsymbol{a_1} + x_2\boldsymbol{a_2} + \cdots + x_n\boldsymbol{a_n}$$

Theorem 3: If A is an $m \times n$ matrix, with columns $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ and if $\mathbf{b}$ is in $\mathbb{R}^n$, the matrix equation

$$A\mathbf{x} = \mathbf{b}$$

has the same solution set as the vector equation

$$x_1\boldsymbol{a_1} + x_2\boldsymbol{a_2} + \cdots + x_n\boldsymbol{a_n} = \mathbf{b}$$

which, in turn, has the same solution set as the system of linear equations whose augmented matrix is

$$[\boldsymbol{a_1}\ \boldsymbol{a_2}\ \ldots\ \boldsymbol{a_n}\ |\ \boldsymbol{b}]$$

Theorem 4: Let A is an $m \times n$ matrix. Then the following statements are **logically equivalent;** that is, for a particular A, either they are all true or they are all false:

a. For each $\mathbf{b}$ in $\mathbb{R}^n$, the equation $A\mathbf{x} = \mathbf{b}$ has a solution.
b. Each $\mathbf{b}$ in $\mathbb{R}^n$ is a linear combination of the columns of A.
c. The columns of A span $\mathbb{R}^n$.
d. A has a pivot position in every row.

Theorem 5: If A is an $m \times n$ matrix, **u** and **v** are vectors in $\mathbb{R}^n$, and c is a scalar, then

$$A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v} \text{ and } A(c\mathbf{u}) = c(A\mathbf{u})$$