MATH 200C Linear Algebra

Highlights from §1.9: Matrix of a Linear Transformations

Definitions: A **transformation** T from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a rule which assigns to each vector **x** in $\mathbb{R}^n$ some vector $T(\mathbf{x})$ in $\mathbb{R}^m$.

Definition: A transformation T from $\mathbb{R}^n$ to $\mathbb{R}^m$ is called a **linear transformation** if for all **u** and **v** in the domain of T and all scalars c, both

(i) $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$, and
(ii) $T(c\,\mathbf{u}) = c\,T(\mathbf{u})$

Theorem: If T is a linear transformation, then

$T(\mathbf{0}) = \mathbf{0}$ and

$T(c\mathbf{u} + d\mathbf{v}) = c\,T(\mathbf{u}) + d\,T(\mathbf{v})$

Theorem: if A is an $m \times n$ matrix, then the transformation $T(\mathbf{x}) = A\mathbf{x}$ is a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$.

BIG QUESTION: If T is a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$, must T be of the form $T(\mathbf{x}) = A\mathbf{x}$ for some $m \times n$ matrix? If so, how do we find A?

Theorem 10: If T is a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$, then there exists a unique matrix A such that $T(\mathbf{x}) = A\mathbf{x}$ for all **x** in $\mathbb{R}^n$.

A is the $m \times n$ matrix whose jth column is $T(\mathbf{e}_j)$ where $\mathbf{e}_j$ is the jth column of the identity matrix in $\mathbb{R}^n$. We call A the **standard matrix** for T.

Definitions: A mapping T from $\mathbb{R}^n$ to $\mathbb{R}^m$ is **onto (surjective)** means for each **b** in $\mathbb{R}^m$, there is **at least one x** in such that $T(\mathbf{x}) = \mathbf{b}$.

A mapping T from $\mathbb{R}^n$ to $\mathbb{R}^m$ is **one-to-one ((injective)** means for each **b** in $\mathbb{R}^m$, there is **at most one x** in such that $T(\mathbf{x}) = \mathbf{b}$.

Theorem 11: Let T: $\mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then T is one-to-one if and only if the equation $T(\mathbf{x}) = \mathbf{0}$ has only the trivial solution.

Theorem 12: Let T: $\mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation with A as its standard matrix. Then

(a) T maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ if and only if the columns of A span $\mathbb{R}^m$.
(b) T is one-to-one if and only if the columns of A are a linearly independent set.