Highlights of Section 2.3: Characterizing Invertible Matrices

Theorem (The Invertible Matrix Theorem): Let A be a square $n \times n$ matrix. Then the following statements are equivalent; that is, for a given matrix A, the statements are either all true or all false.

A is an invertible matrix.

A is row equivalent to the $n \times n$ identity matrix.

A has n pivot positions.

The equation $A\mathbf{x} = \mathbf{b}$ has only the trivial solution.

The columns of A form a linearly independent set.

The linear transformation $\mathbf{x} \rightarrow A\mathbf{x}$ is one – to – one.

The equation $A\mathbf{x} = \mathbf{0}$ has at least one solution for all $\mathbf{b}$ in $\mathbb{R}^n$

The columns of A span $\mathbb{R}^n$.

The linear transformation $\mathbf{x} \rightarrow A\mathbf{x}$ maps $\mathbb{R}^n$onto $\mathbb{R}^n$.

There is an $n \times n$ matrix C such that $CA = I$.

There is an $n \times n$ matrix D such that $AD = I$.

A^{T} is an invertible matrix.

Theorem 9: Let T: $\mathbb{R}^n \rightarrow \mathbb{R}^n$ be a linear transformation and let A be the standard matrix for T. Then T is invertible if and only if A is an invertible matrix. In that case, the linear transformation S given by $S(\mathbf{x}) = A^{-1}\mathbf{x}$ is the unique transformation satisfying

$$S(T(\mathbf{x})) = \mathbf{x} \text{ and } T(S(\mathbf{x})) = \mathbf{x} \text{ for all } \mathbf{x} \text{ in } \mathbb{R}^n.$$