MATH 200C Linear Algebra

Highlights from §2.2: Inverse of a Matrix

Definition: A square $n \times n$ matrix A is **invertible (nonsingular)** if there is an $n \times n$ matrix C such that AC and CA are the
identity $n \times n$ matrix: $AC = CA = I = I_n$ and C is called an **inverse** of A.
Note: There is at most one inverse for a given square matrix A.
Proof: If B and C are each inverses of A, then $B = BI = B(AC) = BA(C) = IC = C$. [Supply a reason to justify each equality].

If A is invertible, the unique inverse of A is denoted A^{-1} .
A non-invertible matrix is also called a **singular** matrix.
Examples of invertible and noninvertible matrices:
The Inverse of $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ is $\begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}$ but the matrix $B = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ is singular.

Theorem 4: Inverses of 2×2 matrices. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $ad - bc \neq 0$, then A is invertible and $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. If $ad - bc = 0$, then A is not invertible.
Theorem 5: If A is an $n \times n$ invertible matrix, then for each **b** in $\mathbb{R}^n$n, the equation A**x** = **b** has the unique solution $\mathbf{x} = A^{-1}\mathbf{b}$
Theorem 6: If A is invertible, then so A^{-1} and $(A^{-1})^{-1} = A$. (The inverse of the inverse is the original matrix)

If A and B are invertible, then so is AB with $(AB)^{-1} = B^{-1} A^{-1}$ (the inverse of the product is the product of the inverses in reverse order)

If A is invertible, then so is A^T with $(A^T)^{-1} = (A^{-1})^T$ (The inverse of the transpose is the transpose of the inverse).

Elementary Matrices

Definition: **An elementary matrix** is a matrix obtained by performing a single elementary row operation on an identity matrix.

If an elementary row operation is performed on an $m \times n$ matrix A, we can write the resulting matrix as EA where the $m \times m$ matrix E is the result of applying the same row operation on I_m . Each elementary matrix E is invertible. The inverse of E is the elementary matrix of the same type that transforms E back into I.

Theorem 7: An $n \times n$ matrix A is invertible if and only if A is row equivalent to I_n. Any sequence of elementary row operations that reduces A to the Identity matrix also transforms the Identity matrix into the inverse of A.

An Algorithm for Finding A^{-1}

Row Reduce the Augmented Matrix [A | I]. If A is row equivalent to I, then [A | I] is row equivalent to [I | A^{-1}]. Otherwise, A does not have an inverse.