MATH 200C Linear Algebra

Highlights of Section 2.1 Matrix Operations

An $m \times n$ **matrix** A is a rectangular array of data with m rows, each of which has n columns. The (i,j)th entry of A, denoted a_{ij} or A_{ij} or $A(i,j)$, is the number in row i and column j of A.
The **diagonal entries** of A are $a_{11}, a_{22}, a_{33}, \ldots$; they form the **main diagonal** of A.
A **diagonal matrix** is a square ($n \times n$) matrix whose nondiagonal entries are all 0.
An $m \times n$ matrix all of whose entries are 0 is called a **zero matrix** and written as 0.
Each column of A is a vector in $\mathbb{R}^m$; we can write the matrix A as $A = [\ \mathbf{a}_1 \quad \mathbf{a}_2 \ldots \mathbf{a}_n\ ]$ where $\mathbf{a}_j$ is the jth column.

Matrix Addition and Scalar Multiplication

Matrix Equality: $A = B$ if both matrices are of the same size and all the corresponding entries are equal; that is, $a_{ij} = b_{ij}$ for all i and j
Matrix Addition: If A and B are $m \times n$ matrices, then the **sum** $A + B$ is the $m \times n$ matrix whose columns are the sums of the corresponding columns are A and B. We have $(A + B)_{ij} = A_{ij} + B_{ij}$ for all i and j.
Scalar Multiplication: If r is a scalar and A is a matrix, then the **scalar multiple** rA is the matrix whose columns are r times the columns of A. We have $(rA)_{ij} = r A_{ij}$ for all i and j.
Theorem 1: Let A, B, and C be matrices of the same size, and let r and s be scalars.

$A + B = B + A$	$r(A + B) = rA + rB$
$(A + B) + C = A + (B + C)$	$(r + s)A = rA + rB$
$A + 0 = A$	$r(sA) = (rs)A$

Matrix Multiplication

Definition: If A is an $m \times n$ matrix and B is an $n \times p$ matrix with columns $\mathbf{b}_1 \quad \mathbf{b}_2 \ldots \mathbf{b}_p$, then the product AB is the $m \times p$ matrix whose columns are $A\mathbf{b}_1, A\mathbf{b}_2, \ldots, A\mathbf{b}_p$; that is $AB = A[\ \mathbf{b}_1 \quad \mathbf{b}_2 \ldots \mathbf{b}_p,\ ] = [\ A\mathbf{b}_1 \quad A\mathbf{b}_2 \ldots A\mathbf{b}_p,\ ]$.
Note: Each column of AB is a linear combination of the columns of A using weights from the corresponding column of B.
Theorem 2: Let A be an $m \times n$ matrix, and let B and C have sizes for which the sums and products are defined.

$A(BC) = (AB)C$	Associative Law of Matrix Multiplication
$A(B + C) = AB + AC$	Left Distributive Law
$(B + C) A = BA + CA$	Right Distributive Law
$r(AB) = (rA)B = A(rB)$	
$I_mA = A = AI_m$	Identity for Matrix Multiplication

WARNING: IN GENERAL $AB \neq BA$

Row-Column Rule for Computing AB

If the product AB is defined, then the entry in row i and column j of AB is the sum of the products of the corresponding entries from row i of A and column j of B:

$$(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + \ldots + a_{in}b_{nj} = \sum_{k=1}^{n} a_{ik}b_{kj}$$

Powers of a Matrix

Definition: If A is a square matrix and k is a positive integer, then A^k is the product of k copies of A

$$\boldsymbol{A^k} = \underbrace{A \cdots A}_{k}$$

We define A^0 to be the identity matrix I.

Transpose of a Matrix

Definition: The **transpose** of a $m \times n$ matrix A, denoted A^T, is the $n \times m$ matrix whose columns are formed from the corresponding rows of A; that is, the entry in row i and column j of A^T is the entry in column j and row i of A.

Theorem 3: Let A and B denote matrices whose sizes are appropriate for the following sums and products

$(A^T)^T = A$	For any scalar r, $(rA)^T = r A^T$
$(A + B)^T = A^T + B^T$	$(AB)^T = B^T A^T$

The transpose of a product of a matrices is the product of their transposes in *reverse* order.