

Determinants

Introduction To Determinants

Key Fact: The **determinant** of a square matrix is a number which is nonzero if and only if the matrix is invertible.

For a 1×1 matrix $A = [a_{11}]$, the determinant of A is the number a_{11} .

For a 2×2 matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, the determinant of A (denoted $\det A$) is defined as the number $a_{11}a_{22} - a_{21}a_{12}$. Easier to remember as $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$.

Notation: For a square matrix A , we let A_{ij} denote the submatrix of A obtained by deleting the i th row and j th column of A .

Definition: For $n \geq 2$, the **determinant of an $n \times n$ matrix** $A = [a_{ij}]$ is the sum of n terms of the form $\pm a_{1j} \det A_{1j}$ with plus and minus signs alternating, where the entries $a_{11}, a_{12}, \dots, a_{1n}$ are from the first row of A :

$$\det A = a_{11} \det A_{11} - a_{12} \det A_{12} + \dots + (-1)^{n+1} a_{1n} \det A_{1n} = \sum_{j=1}^n (-1)^{1+j} a_{1j} \det A_{1j}$$

Definition: The **(i, j) -cofactor** of A is the number $C_{ij} = (-1)^{i+j} \det A_{ij}$.

With this notation, we have $\det A = a_{11}C_{11} + a_{12}C_{12} + \dots + a_{1n}C_{1n}$ which is called the **cofactor expansion across the first row of A**

Theorem 1: The determinant of any $n \times n$ matrix A can be computed by a cofactor expansion across any row or down any column. The expansion across the i th row using the cofactors is

$$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + a_{i3}C_{i3} + \dots + a_{in}C_{in}$$

The cofactor expansion down the j th column is

$$\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + a_{3j}C_{3j} + \dots + a_{nj}C_{nj}$$

The plus or minus sign in the (i, j) -cofactor depends on the position of the number a_{ij} in the matrix, not on the sign of the number a_{ij} . The factor $(-1)^{i+j}$ creates this pattern of signs:

$$\begin{bmatrix} + & - & + & \dots \\ - & + & - & \\ + & - & + & \\ \vdots & & & \ddots \end{bmatrix}$$

Theorem 2: The determinant of a triangular matrix is the product of the diagonal element

Properties of Determinants

Theorem 3: Determinants and the Elementary Row Operations: Let A be square matrix

- (a) If we add a multiple of one row of A to another row to produce a matrix B , then $\det B = \det A$.
- (b) If we interchange two rows of A to produce a matrix B , then $\det B = -\det A$.
- (c) If we multiply one row of A by a constant k to produce a matrix B , then $\det B = k \det A$.

Theorem 4: A square matrix is invertible if and only if its determinant is nonzero. ($\det A \neq 0$)

Theorem 5: The determinant of the transpose of a square matrix equals the determinant of the original matrix ($\det A^T = \det A$).

Theorem 6: Determinant of a product is the product of the determinants: If A and B are $n \times n$ matrices, then $\det (AB) = (\det A)(\det B)$

Determinants as Area or Volume (Read Pages 191 – 195 of Text)

Theorem 9: If A is a 2×2 matrix, the area of the parallelogram determined by the columns of A is $|\det A|$.

If A is a 3×3 matrix, the volume of the parallelepiped determined by the columns of A is $|\det A|$.

Theorem 10: Let $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the linear transformation determined by a 2×2 matrix A . If S is a parallelogram in \mathbb{R}^2 , then Area of $T(S) = |\det A|$ (area of S).

Let $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be the linear transformation determined by a 3×3 matrix A . If S is a parallelepiped in \mathbb{R}^3 , then Volume of $T(S) = |\det A|$ (volume of S).