

## Vector Spaces

### Coordinate Systems

**Theorem 8: The Unique Representation Theorem:** Let  $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$  be a basis for a vector space  $V$ . Then for each  $\mathbf{x}$  in  $V$ , there exists a unique set of scalars  $c_1, \dots, c_n$  such that  $\mathbf{x} = c_1\mathbf{b}_1 + \dots + c_n\mathbf{b}_n$

Definition: Suppose  $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$  is a basis for a vector space  $V$  and  $\mathbf{x}$  is in  $V$ . The **coordinates of  $\mathbf{x}$  relative to the basis  $\mathcal{B}$**  (or the  $\mathcal{B}$ -coordinates of  $\mathbf{x}$ ) are the weights  $c_1, \dots, c_n$  such that  $\mathbf{x} = c_1\mathbf{b}_1 + \dots + c_n\mathbf{b}_n$

If  $c_1, \dots, c_n$  are the  $\mathcal{B}$ -coordinates of  $\mathbf{x}$ , then the vector in  $\mathbb{R}^n$

$$[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

is the **coordinate vector of  $\mathbf{x}$  (relative to  $\mathcal{B}$ )**, or the  **$\mathcal{B}$ -coordinate vector of  $\mathbf{x}$** . The mapping  $\mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{B}}$  is the **coordinate mapping (determined by  $\mathcal{B}$ )**.

### Coordinates in $\mathbb{R}^n$

Let  $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$  be a basis for  $\mathbb{R}^n$ . Let  $P_{\mathcal{B}} = [\mathbf{b}_1 \ \mathbf{b}_2 \ \dots \ \mathbf{b}_n]$ . Then the vector equation  $\mathbf{x} = c_1\mathbf{b}_1 + \dots + c_n\mathbf{b}_n$  is equivalent to  $\mathbf{x} = P_{\mathcal{B}}[\mathbf{x}]_{\mathcal{B}}$

We call  $P_{\mathcal{B}}$  the **change-of-coordinates matrix** from  $\mathcal{B}$  to the standard basis in  $\mathbb{R}^n$ . Left-multiplication by  $P_{\mathcal{B}}$  transforms the coordinate vector  $[\mathbf{x}]_{\mathcal{B}}$  into  $\mathbf{x}$ .

**Theorem 9:** Let  $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$  be a basis for a vector space  $V$ . Then the coordinate mapping  $\mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{B}}$  is a one-to-one linear transformation from  $V$  onto  $\mathbb{R}^n$ .

The coordinate mapping in Theorem 9 is an important example of an **isomorphism**  $V$  onto  $\mathbb{R}^n$ . In general, a one-to-one linear transformation from a vector space  $V$  onto a vector space  $W$  is called an isomorphism from  $V$  onto  $W$  (*iso* from the Greek for “the same,” and *morph* from the Greek for “form” or “structure”). The notation and terminology for  $V$  and  $W$  may differ, but the two spaces are indistinguishable as vector spaces. Every vector space calculation in  $V$  is accurately reproduced in  $W$ , and vice versa. In particular, any real vector space with a basis of  $n$  vectors is indistinguishable from  $\mathbb{R}^n$ . (From Lay, *Linear Algebra*).