

Vector Spaces

Dimension of a Vector Space

Theorem 10: If a vector space V has a basis $B = \{ \mathbf{b}_1, \dots, \mathbf{b}_n \}$, then any set in V containing more than n vectors must be linearly dependent.

Theorem 10 also applies to infinite sets in V . An infinite set is said to be linearly dependent if some finite subset is linearly dependent; otherwise, the set is linearly independent. If S is an infinite set in V , take any subset $\{ u_1, \dots, u_p \}$ of S , with $p > n$. The proof above shows that this subset is linearly dependent and hence so is S ."

Theorem 11: If a vector space V has a basis of n vectors, then every basis of V must consist of exactly n vectors."

Definition: If a vector space V is spanned by a finite set, then V is said to be **finite-dimensional**, and the **dimension** of V , written as **dim** V , is the number of vectors in a basis for V . The dimension of the zero vector space $\{0\}$ is defined to be zero. If V is not spanned by a finite set, then V is said to be **infinite-dimensional**.

Theorem 12: Let H be a subspace of a finite-dimensional vector space V . Any linearly independent set in H can be expanded, if necessary, to a basis for H . Also, H is finite-dimensional and $\dim H \leq \dim V$

Theorem 13 The Basis Theorem Let V be a p -dimensional vector space, $p \geq 1$. Any linearly independent set of exactly p elements in V is automatically a basis for V . Any set of exactly p elements that spans V is automatically a basis for V .

The Dimensions of Nul A , Col A , and Row A

Definition: The **rank** of an $m \times n$ matrix A is the dimension of the column space and the nullity of A is the dimension of the null space."

Theorem 14 The Rank Theorem The dimensions of the column space and the null space of an $m \times n$ matrix A satisfy the equation $\text{rank } A + \text{nullity } A = \text{number of columns in } A$

"Rank and the Invertible Matrix Theorem"

Theorem The Invertible Matrix Theorem (continued) Let A be an $n \times n$ matrix. Then the following statements are each equivalent to the statement that A is an invertible matrix.

The columns of A form a basis of \mathbb{R}^n

$$\text{Col } A = \mathbb{R}^n$$

$$\text{rank } A = n$$

$$\text{nullity } A = 0$$

$$\text{Nul } A = \{0\}$$