MATH 200C Linear Algebra

Vector Spaces

Linearly Independent Sets; Basis

Definition: An indexed set of vectors $\{\mathbf{v}_1,\ldots,\mathbf{v}_p\}$ in a vector space V is **linearly independent** if the equation (1) $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p = \mathbf{0}$ has only the trivial solution, $c_1 = 0, \ldots, c_p = 0$

The set $\{\mathbf{v}_1,\ldots,\mathbf{v}_p\}$ is said to be **linearly dependent** if (1) has a nontrivial solution, that is, if there are some weights, $c_1,\ldots,c_p$, not all zero, such that (1) holds. In such a case, (1) is called a **linear dependence relation** among $\mathbf{v}_1,\ldots,\mathbf{v}_p$.

Theorem 4: An indexed set $\{\mathbf{v}_1,\ldots,\mathbf{v}_p\}$ of two or more vectors, with $\mathbf{v}_1 \neq \mathbf{0}$, is linearly dependent if and only if some $\mathbf{v}_j$ (with $j > 1$) is a linear combination of the preceding vectors, $\mathbf{v}_1,\ldots,\mathbf{v}_{j-1}$.

Definition: Let H be a subspace of a vector space V. A set of vectors B in V is a **basis** for H if

B is a linearly independent set, and the subspace spanned by B coincides with H; that is,

$H = \text{Span } B$.

Theorem 5 The Spanning Set Theorem Let $S = \{\mathbf{v}_1,\ldots,\mathbf{v}_p\}$ be a set in a vector space V, and let $H = \text{Span}\{\mathbf{v}_1,\ldots,\mathbf{v}_p\}$. If one of the vectors in S—say, $\mathbf{v}_k$—is a linear combination of the remaining vectors in S, then the set formed from S by removing $\mathbf{v}_k$ still spans H.

If $H \neq \{0\}$, some subset of S is a basis for H.

Theorem 6: The pivot columns of a matrix A form a basis for Col A

Theorem 7: If two matrices A and B are row equivalent, then their row spaces are the same. If B is in echelon form, the nonzero rows of B form a basis for the row space of A as well as for that of B.