

## Vector Spaces

### *Null Spaces, Column Spaces, Row Spaces, and Linear Transformations*

Definition: The **null space** of an  $m \times n$  matrix  $A$ , written as  $\text{Nul } A$ , is the set of all solutions of the homogeneous equation  $A\mathbf{x} = \mathbf{0}$ :  $\text{Nul } A = \{ \mathbf{x} : \mathbf{x} \text{ is in } \mathbb{R}^n \text{ and } A\mathbf{x} = \mathbf{0} \}$ .

**Theorem 1:** The null space of an  $m \times n$  matrix  $A$  is a subspace of  $\mathbb{R}^n$ . Equivalently, the set of all solutions to a system  $A\mathbf{x} = \mathbf{0}$  of  $m$  homogenous equations in  $n$  unknowns is a subspace of  $\mathbb{R}^n$ .

Definition: The **column space** of an  $m \times n$  matrix  $A$ , written as  $\text{Col } A$ , is the set of all linear combinations of the columns of  $A$ . If  $A = [\mathbf{a}_1 \dots \mathbf{a}_n]$ , then  $\text{Col } A = \text{Span}\{\mathbf{a}_1 \dots \mathbf{a}_n\}$ .

**Theorem 2:** The column space of an  $m \times n$  matrix  $A$  is a subspace of  $\mathbb{R}^m$ . The column space of an  $m \times n$  matrix  $A$  is all of  $\mathbb{R}^m$  if and only if the equation  $A\mathbf{x} = \mathbf{b}$  has a solution for all  $\mathbf{b}$  in  $\mathbb{R}^m$ .

Definition: The **row space** of an  $m \times n$  matrix  $A$  is the set of all linear combinations of the rows of  $A$ .

**Theorem:** The row space of an  $m \times n$  matrix  $A$  is a subspace of  $\mathbb{R}^n$  and is equal to the column space of the transpose of  $A$ .

Definition: A **linear transformation  $T$  from a vector space  $V$  into a vector space  $W$**  is a rule that assigns to each vector  $\mathbf{x}$  in  $V$  a unique vector  $T(\mathbf{x})$  in  $W$  such that

$$T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}) \text{ for all } \mathbf{u}, \mathbf{v} \text{ in } V, \text{ and}$$

$$T(c\mathbf{u}) = c T(\mathbf{u}) \text{ for all } \mathbf{u} \text{ in } V \text{ and all scalars } c.$$

### Nul $A$

1. Nul  $A$  is a subspace of  $\mathbf{R}^n$ .
2. Nul  $A$  is implicitly defined; that is, you are given only a condition  $A\mathbf{x} = \mathbf{0}$  that vectors in Nul  $A$  must satisfy.
3. It takes time to find vectors in Nul  $A$ . Row operations on  $\begin{bmatrix} A & \mathbf{0} \end{bmatrix}$  are required.
4. There is no obvious relation between Nul  $A$  and the entries in  $A$ .
5. A typical vector  $\mathbf{v}$  in Nul  $A$  has the property that  $A\mathbf{v} = \mathbf{0}$ .
6. Given a specific vector  $\mathbf{v}$ , it is easy to tell if  $\mathbf{v}$  is in Nul  $A$ . Just compute  $A\mathbf{v}$ .
7. Nul  $A = \{\mathbf{0}\}$  if and only if the equation  $A\mathbf{x} = \mathbf{0}$  has only the trivial solution.
8. Nul  $A = \{\mathbf{0}\}$  if and only if the linear transformation  $\mathbf{x} \mapsto A\mathbf{x}$  is one-to-one.

### Col $A$

1. Col  $A$  is a subspace of  $\mathbf{R}^m$ .
2. Col  $A$  is explicitly defined; that is, you are told how to build vectors in Col  $A$ .
3. It is easy to find vectors in Col  $A$ . The columns of  $A$  are displayed; others are formed from them.
4. There is an obvious relation between Col  $A$  and the entries in  $A$ , since each column of  $A$  is in Col  $A$ .
5. A typical vector  $\mathbf{v}$  in Col  $A$  has the property that the equation  $A\mathbf{x} = \mathbf{v}$  is consistent.
6. Given a specific vector  $\mathbf{v}$ , it may take time to tell if  $\mathbf{v}$  is in Col  $A$ . Row operations on  $\begin{bmatrix} A & \mathbf{v} \end{bmatrix}$  are required.
7. Col  $A = \mathbf{R}^m$  if and only if the equation  $A\mathbf{x} = \mathbf{b}$  has a solution for every  $\mathbf{b}$  in  $\mathbf{R}^m$ .
8. Col  $A = \mathbf{R}^m$  if and only if the linear transformation  $\mathbf{x} \mapsto A\mathbf{x}$  maps  $\mathbf{R}^n$  onto  $\mathbf{R}^m$ .