MATH 200C *Notes on Sample Examination 2*

(I'll write vectors as rows rather than columns to save space)

1. Let R^4 be the usual Euclidean 4-dimensional space.

(a) List the elements of a 5 element subset of R^4 which spans R^4 but is not linearly independent**: (1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1) and any other 4 dimensional vector**.

(b) List the members of a 3-element subset of R^4 which is linearly independent but does not span R^4**. 1,0,0,0), (0,1,0,0), (0,0,1,0),**

(c) List the members of a nonempty subset of R^4 which is neither linearly independent nor spanning. **{ (1,0,0,0), (2,0,0,0)}**

(d) List the members of a subset of R^4 which is a basis for R^4. **(1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1)**

2. Short answer Questions:

(a) Suppose A is a 9 x 9 matrix such that $A^3 = A$. What the possible values for the determinant of A? **If x = det A, then det(A^3) = x^3 since the determinant of a product is the product of the determinants. Hence $x^3=x$ so $x(x^2-1) = 0$ and hence $x = 0$ or 1 or -1.**

(b)If A and B are invertible 11 x 11 matrices, What is the inverse of ABA^{-1}? $\boldsymbol{AB^{-1}A^{-1}}$

(c) A a vector space with no finite basis: **The space of all polynomials**

(d) What is the dimension of the vector space of all 3×2 matrices? **6**

(e) If A is a 7×9 matrix with a two dimensional null space, what is the rank of A? **rank A + dim Nul A = 9 so rank A + 2 = 9; hence rank A = 7**

(f) Can a 6×9 matrix B have a two-dimensional null space? Explain. **No. If dim Nul B = 2, then B has rank 7 by the Rank Theorem. But the columns of B are vectors in R^6 and so the column space of B cannot exceed 6.**

(g) What is the dimension of the vector space of all polynomials of degree ≤ 12? **13**

3. A personalized question. Let b_1 = the number of the month in which you were born (1 = January, 2 = February, etc..), b_2 = the day of the month on which you were born, and b_3 = the year of your birth. Form the vector $\mathbf{b} = [\mathbf{b_1}\ \ \mathbf{b_2}\ \ \mathbf{b_3}]$ Suppose A is any 3×3 matrix such that the equation A**x** = **b** has a unique solution. Explain why A must be invertible. **If *A* is not invertible, then the system Ax = b either has no solutions or infinitely many solutions since the redcued row echelon form of *A* will have at least one row of all zero's.**

4. (a) Let $S = \{\mathbf{w_1}, \mathbf{w_2}, \mathbf{w_3}\}$ be a linearly independent subset of a vector space V and suppose **w** is a member of V which is not in the span of S. Show that $\{\mathbf{w_1}, \mathbf{w_2}, \mathbf{w_3}, \mathbf{w}\}$ is a linearly independent set.

Suppose $k_1 \mathbf{w_1}, + k_2 \mathbf{w_2} + k_3 \mathbf{w_3} + k_4 \mathbf{w} = 0$ for constants k_1, k_2, k_3, k_4.

If $k_4 = 0$, then $k_1 \mathbf{w_1}, + k_2 \mathbf{w_2} + k_3 \mathbf{w_3} = 0$ but since = $\{\mathbf{w_1}, \mathbf{w_2}, \mathbf{w_3}\}$ is linearly independent, we would have $k_1 = k_2 = k_3 = 0$ and this would imply that $\{\mathbf{w_1}, \mathbf{w_2}, \mathbf{w_3}, \mathbf{w}\}$ is linearly independent since the only linear combination of the vectors equal to the zero vector is the trivial combination.

Hence we must have $k_4 \neq 0$ in which case we have $k_4\mathbf{w} = -k_1 \mathbf{w_1} - k_2 \mathbf{w_2} - k_3 \mathbf{w_3}$ so

$\mathbf{w}_4 = -\left(\frac{k_1}{k_4}\right)\mathbf{w}_1 - \left(\frac{k_2}{k_4}\right)\mathbf{w}_2 - \left(\frac{k_3}{k_4}\right)\mathbf{w}_3$; that is, **w** is a linear combination of $\mathbf{w_1}\ \mathbf{w_2}, \mathbf{w_3}$which means **w** is in the span of S, which is a contradiction.

5. Let T be a linear transformation from R^2 to R^2

(a) Show that the set S of vectors **X** in R^2 such that T (**X**) = 3**X** is a subspace of R^2.

Suppose **X** and **Y** are members of S and k is any scalar. Then $T(k\mathbf{X}) = k\,T(\mathbf{X}) = k\,3\mathbf{X}$ [since **X** belongs to S] = $3\,k\mathbf{X}$ [scalar multiplication of reals is commutative]. Thus $k\mathbf{X}$ is also in S.

Moreover, $T(\mathbf{X} + \mathbf{Y}) = T(\mathbf{X}) + T(\mathbf{Y}) = 3\mathbf{X} + 3\mathbf{Y} = 3(\mathbf{X} + \mathbf{Y})$ [distributive law] and hence **X** + **Y** is in S as well. Therefore S is a subspace.

(b) If T is the particular matrix transformation for which $T(\mathbf{e_1}) = (3,8)$ and $T(\mathbf{e_2}) = (0,-1)$, show that $T(\mathbf{X}) = 3\mathbf{X}$ if $\mathbf{X} = (1,2)$. $T(\mathbf{X}) = T((1,2)) = T(1\,\mathbf{e_1} + 2\,\mathbf{e_2}) = 1\,T(\mathbf{e_1}) + 2\,T(\mathbf{e_2}) = (3,8) + 2\,(0,-1) = (3,8) + (0,-2) = (3+0, 8-2) = (3,6) = 3\,(1,2) = 3\,\mathbf{X}$.

6. The reduced row echelon form of $A = \begin{bmatrix} 7 & 21 & 4 & 112 & 11 \\ 12 & 37 & 3 & 170 & -16 \\ -4 & -2 & -2 & -62 & -4 \\ 16 & 48 & 12 & 276 & 48 \end{bmatrix}$ is $R = \begin{bmatrix} 1 & 0 & 0 & -3 & 9 \\ 0 & 1 & 0 & 5 & -4 \\ 0 & 0 & 1 & 7 & 8 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Find each of the following:

(a) Number of pivot columns of *A: 3*

(b) Number of pivot positions of *R: 3*

(c) rank of A: 3

(d) Dimension of the column space of A: 3

(e) nullity of A: 2 (number of columns of *R* with no leading 1's;also equals number of columns – rank)

(f) nullity of the transpose of *A*: 1 (rank *A* + nullity A^T = number of rows of *A*)

(g) a basis for the row space of *A* { (1,0,0,-3,9), (0,1,0,5,-4), {0,0,1,7,8)} (the rows of *R* with leading 1's)

(h) a basis for the column space of *A*: First three columns of *A* (The columns of *A* corresponding to the columns of *R* with leading 1's)

(i) Describe a procedure (you do not have to carry out the computations) for finding a basis for the row space which is made up of some of the rows of *A*. Reduce the transpose of *A* to row echelon form *R*. Pick the columns of A^T which corresponds to the columns of *R* with leading 1's. The transposes of those columns form a basis for the row space of *A*.

7. Let *S* be the parallelogram determined by the vectors $\mathbf{b}_1 = [4, -7]^T$ and $\mathbf{b}_2 = [0,1]^T$ and let A be the matrix $\begin{bmatrix} 7 & 2 \\ 1 & 1 \end{bmatrix}$. Compute the area of the image of *S* under the linear transformation whose matrix representation is *A.*

S is the image of the standard unit square *U* = (vertices (0,0), (1,0), (0,1), (1,1)) under the linear transformation with matrix $B = \begin{bmatrix} 4 & 0 \\ -7 & 1 \end{bmatrix}$. Then the image of *S* under *A* is the image of U under the transformation with matrix representative BA. Thus its area is | det (BA)| = | det B| |det A| = 4*5=20.

8. (Leontief Input-Output Model). Consider the production model $\mathbf{x} = C\mathbf{x} + \mathbf{d}$ for an economy with two sectors where $C = \begin{bmatrix} .2 & .5 \\ .6 & .1 \end{bmatrix}, \mathbf{d} = \begin{bmatrix} 112 \\ 84 \end{bmatrix}$. Determine the production level vector necessary to satisfy the final demand.

Write the consumption matrix C as $\begin{bmatrix} \frac{1}{5} & \frac{1}{2} \\ \frac{3}{5} & \frac{1}{10} \end{bmatrix}$. Then I – C = $\begin{bmatrix} \frac{4}{5} & -\frac{1}{2} \\ -\frac{3}{5} & \frac{9}{10} \end{bmatrix}$ so $(I - C)^{-1}$ =

$\frac{1}{21}\begin{bmatrix} 45 & 25 \\ 30 & 40 \end{bmatrix}$ and the production vector is **x** = $\frac{1}{21}\begin{bmatrix} 45 & 25 \\ 30 & 40 \end{bmatrix}\begin{bmatrix} 112 \\ 84 \end{bmatrix} = \begin{bmatrix} 340 \\ 320 \end{bmatrix}$