Some Very Brief Notes on Sample Exam 3

These are quick answers, most of which would need further explanation and display of intermediate work to earn full credit

Question 1: Concepts

a. A vector space is a set V with operations of addition and scalar multiplication satisfying: closure, associativity, commutativity of addition, additive identity, additive inverse, distributive properties, and scalar identity.

b. A subset $W \subseteq V$ is a subspace if $\mathbf{0} \in W$, W is closed under addition and W is closed under scalar mulltiplcation.

c. Define 2 of these 3 terms:
Linear independence: Only trivial solution exists for $c_1v_1 + \cdots + c_nv_n = 0$
Basis: Linearly independent set that spans
Dimension: Number of vectors in a basis

d. A matrix is diagonalizable if $A = PDP^{-1}$ for some diagonal matrix D.

Question 2: True/False

a. False: $\{(1,1,1),(2,2,2),(3,3,3)\}$ is linearly dependent
b. False: It also needs to be linearly independent. $\{(1,0),(0,1),(1,1)\}$ spans R^2 but is not a basis.
c. True if matrix is 3×3 but could be false for a larger square matrix.
d. False: eigenvectors are defined to be nonzero vectors.
e. True: See Theorem 4 of Section 5.2

Question 3: Computations

a : W contains zero vector and is closed under addition and scalar multiplication so it is a subspace.
b. $\mathbf{v}_2 = 2\mathbf{v}_1$ so set is linearly dependent.
c. $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 1 & 3 & 1 \end{bmatrix}$ has rank 2. One basis is $\{(1,0,1),(2,1,3)\}$ so dimension is 2.
d. $(3,1) = c_1(1,1) + c_2(1,-1)$ yields the system $\begin{matrix} c_1 + c_2 = 3 \\ c_1 - c_2 = 1 \end{matrix}$ which has the unique solution$c_1 = 2, c_2 = 1$.
Coordinate vector is $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Question 4: Eigenvalues and Eigenvectors

a. $\det(A - \lambda I) = (2-\lambda)^2 - 1 = \lambda^2 - 4\lambda + 3 = (\lambda - 3)(\lambda = 1)$; eigenvalues are $\lambda_1 = 3, \lambda_2 = 1$. For $\lambda_1 = 3$, $\mathbf{v_1} = (1,1)$; For $\lambda_2 = 1$, $\mathbf{v_2} = (1,-1)$
b. The matrix $A = \begin{bmatrix} 4 & 1 \\ 0 & 4 \end{bmatrix}$ has only one eigenvalue $\lambda = 4$ whose eigenvalues are nonzero multiples of (1,0) so A is not diagonalizable.
c. $A = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}$ is already a diagonal matrix so we can choose $P = I$.

Question 5: Proofs

a. Any two bases have the same number of elements by the exchange theorem.
b. Eigenvectors corresponding to distinct eigenvalues are linearly independent; see Theorem 2 of Section 5.2.

Question 6: Application

a Use $A^k = PD^kP^{-1}$ to begin your argument.
b Need to know that eigenvalue of algebraic multiplicity 2 also has geometric multiplicity 2; that the eigenvector associated with $\lambda = 2$ has a pair of linearly independent eigenvectors.
c The unique steady-state stochastic vector is $(\frac{1}{3}, \frac{2}{3})$ so we expect 2/3 to be Android users.