1. The reduced row echelon form of the matrix A is $B = \begin{bmatrix} 1 & 0 & 0 & -\dfrac{1}{2} & -2 \\ 0 & 1 & 0 & -\dfrac{1}{10} & \dfrac{1}{5} \\ 0 & 0 & 1 & -\dfrac{29}{10} & \dfrac{4}{5} \end{bmatrix}$

(a) From B, we see that $x = (x_1, x_2, x_3, x_4, x_5)$ is in the null space when $x_1 = \dfrac{1}{2}x_4 + 2x_5$, $x_2 = \dfrac{1}{10}x_4 - \dfrac{1}{5}x_5$,

$x_3 = \dfrac{29}{10}x_4 - \dfrac{4}{5}x_5$, $x_4 =$ any value s and $x_5 =$ any value t. Thus vectors in the null space have the form

$$\left(\left(\dfrac{1}{2}\right)s + 2t, \dfrac{1}{10}s - \dfrac{1}{5}s, \dfrac{29}{10}s - \dfrac{4}{5}t, s, t\right) = \left(\left(\dfrac{1}{2}\right)s, \dfrac{1}{10}s, \dfrac{29}{10}s, s, 0\right) + \left(2t, -\dfrac{1}{5}t, -\dfrac{4}{5}t, 0, t\right) =$$

$s\left(\dfrac{1}{2}, \dfrac{1}{10}, \dfrac{29}{10}, 1, 0\right) + t\left(2, -\dfrac{1}{5}, -\dfrac{4}{5}, 0, 1\right)$ so a basis for the null space is $\left\{\left(\dfrac{1}{2}, \dfrac{1}{10}, \dfrac{29}{10}, 1, 0\right), \left(2, -\dfrac{1}{5}, -\dfrac{4}{5}, 0, 1\right)\right\}$

(b) Since rank + nullity = 5 and nullity = 2, the rank of A = dimension of column space of A is 3. But the column space for this A is a subspace of R^3 so in this case, the column space is all of R^3 so any linearly independent set of three vectors in R^3 is a basis. We could choose the standard basis $\{ e_1, e_2, e_3 \}$, for example, or the first three columns of A.

2. Show that there is a unique linear transformation T from R^2 to R^2 such that $T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and

$T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$

Since the set $\{ u_1, u_2 \}$ $\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$ is linearly independent, it forms a basis for R^2 and hence each vector v in R^2

has a unique representation $v = k_1 u_1 + k_2 u_2$ as a linear combination of u_1 and u_2. Then the function $T(v) =$

$k_1\begin{bmatrix} 2 \\ 3 \end{bmatrix} + k_2\begin{bmatrix} 1 \\ 4 \end{bmatrix}$ is a linear transformation which has $T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$. Furthermore, any

linear transformation which has $T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$ must also satisfy $T(v) = T(k_1 u_1 + k_2 u_2) =$

$k_1 T(u_1) + k_2 T(u_2) = k_1\begin{bmatrix} 2 \\ 3 \end{bmatrix} + k_2\begin{bmatrix} 1 \\ 4 \end{bmatrix}$ so it is unique.

3. Show that the set $\{x^3, 3x^2 - 6x, 6x - 12, 6\}$ is a basis for P$_3$.

The zero vector in P$_3$ is the polynomial $0x^3 + 0x^2 + 0x + 0$. Suppose there are constants a,b,c,d such that $ax^3 + b(3x^2-6x) + c(6x - 12) + d(6) = 0x^3 + 0x^2 + 0x + 0$. The left hand side equals $ax^3 + 3bx^2 + (6c - 6b)x + (6d - 12c)$. For this polynomial to equal the zero polynomial, we need $a = 0$, $3b = 0$, $6c - 6b = 0$ and $6d - 12c = 0$. From the first two equations, we have $a = 0$ and $b = 0$. Then $6c = 6b = 6(0) = 0$ so $c = 0$. Finally, $6d = 12c = 12(0) = 0$ makes $d = 0$. Thus the only linear combination of the 4 elements equal to **0** is the trivial one with $a = b = c = d = 0$. Thus the set is linearly independent. Since there are four vectors in the set and P$_3$ has dimension 4, any linearly independent set of 4 vectors is a basis.

4. (a) Suppose A**v** = **b** and **h** is in the null space of A. Then A**h** = 0 and A(**v** + **h**) = A**v** + A**h** (since matrix multiplication satisfies the distributive law) but A**v** = **b** and A**h** = **0** so A(**v** + **h**) = A**v** + A**h** = **b** + **0** = **0**, the last equality following by the properties of the zero vector in a vector space.

   (b) Let **w** be any other solution to A**v** = **b** and consider the vector **h** = **w** – **x**. Then A**h** = A(**w** – **x**) = A**w** – A**x** = **b** – **b** = 0. Thus **h** is in the null space. Thus **w** = **x** + **h** is the sum of the particular solution **x** and a vector **h** in the null space.

   (c) The row operations used to reduce A we used in Problem 1 will transform the vector $\begin{bmatrix} 3 \\ 7 \\ 11 \end{bmatrix}$ into

$\begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix}$ so a particular solution is **x** = (5, 0, 1, 0, 0). From Problem 1, **h** = $s\left(\dfrac{1}{2},\dfrac{1}{10},\dfrac{29}{10},1,0\right) + t\left(2,-\dfrac{1}{5},-\dfrac{4}{5},0,1\right)$

5.   (a) If A and B are any pair of elements in V and k is any scalar, then we have T(A +B) = C(A + B) = CA +CB = T(A) + T(B) and T(kA) = C(kA) = k(CA) = kT(A) so T is a linear transformation for any 2×2 matrix C.

This particular matrix C does not produce an isomorphism since the nonzero vector $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ is in the kernel of

T; an isomorphism only has the zero vector in its kernel.

   (b) Find the matrix for T relative to the standard basis for V. The standard basis is

$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ which T sends to $\left\{ \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 0 & 2 \end{bmatrix} \right\}$. Using the standard

isomorphism from the 2×2 matrices to R4, the matrix representing the transformation can be written as

$\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix}$

6. (a) Let $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\mathbf{u}_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$. Then $\mathbf{v}_1 = \mathbf{u}_1$. Now $<\mathbf{u}_2, \mathbf{v}_1> = \mathbf{u}_2 . \mathbf{v}_1 = 1$ -1 + 1 = 1 and $||\mathbf{v}_1||^2 = 3$ so $\mathbf{v}_2 =$

$\mathbf{u}_2 - \dfrac{1}{3}\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} - \dfrac{1}{3}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} \\ -\frac{4}{3} \\ \frac{2}{3} \end{bmatrix}$, so an orthogonal basis is $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} \frac{2}{3} \\ -\frac{4}{3} \\ \frac{2}{3} \end{bmatrix} \right\}$. To generate an orthonormal basis,

we need to divide each of these vectors by its length. Here $||\mathbf{v_2}|| = \sqrt{\dfrac{4}{9} + \dfrac{16}{9} + \dfrac{4}{9}} = \sqrt{\dfrac{24}{9}} = \dfrac{2}{3}\sqrt{6}$ so the

orthonormal basis is $\left\{ \begin{bmatrix} \dfrac{1}{\sqrt{3}} \\ \dfrac{1}{\sqrt{3}} \\ \dfrac{1}{\sqrt{3}} \end{bmatrix}, \begin{bmatrix} \dfrac{1}{\sqrt{6}} \\ \dfrac{-2}{\sqrt{6}} \\ \dfrac{1}{\sqrt{6}} \end{bmatrix} \right\}$

(b) $\text{proj}_W \mathbf{u} = <\mathbf{u}, \mathbf{v_1}> \mathbf{v_1} + <\mathbf{u}, \mathbf{v_2}> \mathbf{v_2}$. With $\begin{bmatrix} 12 \\ 8 \\ 16 \end{bmatrix}$, we have $< \mathbf{u}, \mathbf{v_1} > = \dfrac{36}{\sqrt{3}}$ and $<\mathbf{u}, \mathbf{v_2}> = \dfrac{12}{\sqrt{3}}$ so $\text{proj}_W \mathbf{u} =$

$\begin{bmatrix} 12 \\ 12 \\ 12 \end{bmatrix} + \begin{bmatrix} 2\sqrt{2} \\ -4\sqrt{2} \\ 2\sqrt{2} \end{bmatrix} = \begin{bmatrix} 12+2\sqrt{2} \\ 12-4\sqrt{2} \\ 12+2\sqrt{2} \end{bmatrix}$

7. (a) Since A and B are similar, there is an invertible matrix P such that $P^{-1}AP = B$. Then $B^2 = BB = (P^{-1}AP)(P^{-1}AP) = P^{-1}A(PP^{-1})AP = P^{-1}AIAP = P^{-1}AAP = P^{-1}A^2P$ so B^2 is similar to A^2.

(b) Since A = CD and C is invertible, we have $C^{-1}A = C^{-1}(CD) = (C^{-1}C)D = ID = D$. Now multiply $C^{-1}A = D$ on the right by C and we have $C^{-1}AC = DC$; that is, A is similar to DC.

8. Since v_i is an eigenvector of A associated with the eigenvalue λ for each i, we have $Av_i = \lambda v_i$. If $\mathbf{v}$ is any linear combination of $v_1, v_2, ...v_k$ then $v = a_1v_1 + a_2v_2 + ...+ a_kv_k$ for some scalars $a_1, a_2,..,a_k$. Then
$Av = A(a_1v_1 + a_2v_2 + ...+ a_kv_k) = Aa_1v_1 + Aa_2v_2 + ...+ Aa_kv_k = a_1Av_1 + a_2Av_2 + ...+ a_kAv_k = a_1\lambda v_1 + a_2\lambda v_2 + ...+ a_k\lambda v_k = \lambda(a_1v_1 + a_2v_2 + ...+ a_kv_k) = \lambda v$. Thus $\mathbf{v}$ is also an eigenvector associated with λ.

9. We seek a and b so that the line $y = a + bx$ best approximates the data.

Let A = $\begin{bmatrix} 1 & 12 \\ 1 & 23 \\ 1 & 30 \\ 1 & 38 \\ 1 & 47 \\ 1 & 59 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 11 \\ 21 \\ 31 \\ 40 \\ 49 \\ 59 \end{bmatrix}$. Then $A^TA = \begin{bmatrix} 6 & 209 \\ 209 & 8707 \end{bmatrix}$ and $A^T\mathbf{b} = \begin{bmatrix} 211 \\ 8849 \end{bmatrix}$

The solution of $A^TA\,\mathbf{x} = A^T\mathbf{b} = \begin{bmatrix} -1752/1223 \\ 1285/1223 \end{bmatrix}$. Thus $a = -\dfrac{1752}{1223}$, $b = \dfrac{1285}{1223}$