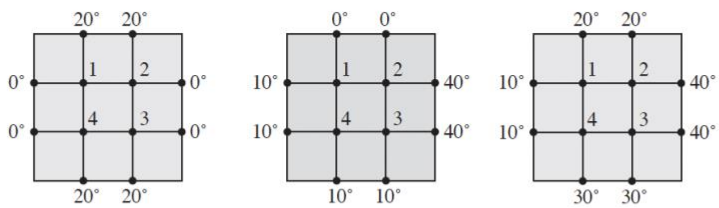
MATH 200C: Linear Algebra

Some Notes on Assignment 12

§ 1.10: 14; § 2.1: 10, 12, 15, 17, 20 , 23; § 2.2: 1, 5, 7, 9

Problem from §1.10

14. Here are Figs. (a) and (b) for Exercise 14, followed by the figure for Exercise 43 in Section 1.1:



(a) (b) Section 1.1

For Fig. (a), the equations are

$$\begin{aligned} 4T_1 &= 0 + 20 + T_2 + T_4 \\ 4T_2 &= T_1 + 20 + 0 + T_3 \\ 4T_3 &= T_4 + T_2 + 0 + 20 \\ 4T_4 &= 0 + T_1 + T_3 + 20 \end{aligned}$$

To solve the system, rearrange the equations and row reduce the augmented matrix. Interchanging rows 1 and 4 speeds up the calculations. The first five steps are shown in detail.

$$\begin{bmatrix} 4 & -1 & 0 & -1 & 20 \\ -1 & 4 & -1 & 0 & 20 \\ 0 & -1 & 4 & -1 & 20 \\ -1 & 0 & -1 & 4 & 20 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & -4 & -20 \\ -1 & 4 & -1 & 0 & 20 \\ 0 & -1 & 4 & -1 & 20 \\ 4 & -1 & 0 & -1 & 20 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & -4 & -20 \\ 0 & 4 & 0 & -4 & 0 \\ 0 & -1 & 4 & -1 & 20 \\ 0 & -1 & -4 & 15 & 100 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & -4 & -20 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & -1 & 4 & -1 & 20 \\ 0 & -1 & -4 & 15 & 100 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 1 & -4 & -20 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 4 & -2 & 20 \\ 0 & 0 & -4 & 14 & 100 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & -4 & -20 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 4 & -2 & 20 \\ 0 & 0 & 0 & 12 & 120 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 10 \\ 0 & 1 & 0 & 0 & 10 \\ 0 & 0 & 1 & 0 & 10 \\ 0 & 0 & 0 & 1 & 10 \end{bmatrix}$$

For Fig (b), the equations are

$$\begin{aligned} 4T_1 &= 10 + 0 + T_2 + T_4 \\ 4T_2 &= T_1 + 0 + 40 + T_3 \\ 4T_3 &= T_4 + T_2 + 40 + 10 \\ 4T_4 &= 10 + T_1 + T_3 + 10 \end{aligned}$$

Rearrange the equations and row reduce the augmented matrix:

$$\begin{bmatrix} 4 & -1 & 0 & -1 & 10 \\ -1 & 4 & -1 & 0 & 40 \\ 0 & -1 & 4 & -1 & 50 \\ -1 & 0 & -1 & 4 & 20 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 10 \\ 0 & 1 & 0 & 0 & 17.5 \\ 0 & 0 & 1 & 0 & 20 \\ 0 & 0 & 0 & 1 & 12.5 \end{bmatrix}$$

a. Here are the solution temperatures for the three problems studied:

Fig. (a) in Exercise 14 of Section 1.10:	(10, 10, 10, 10)
Fig. (b) in Exercise 14 of Section 1.10:	(10, 17.5, 20, 12.5)
Figure for Exercises 43 in Section 1.1	(20, 27.5, 30, 22.5)

When the solutions are arranged this way, it is evident that the third solution is the sum of the first two solutions. What might not be so evident is that the list of boundary temperatures of the third problem is the sum of the lists of boundary temperatures of the first two problems. (The temperatures are listed clockwise, starting at the left of T_1.)

Fig. (a):	(0, 20, 20, 0, 0, 20, 20, 0)
Fig. (b):	(10, 0, 0, 40, 40, 10, 10, 10)
Fig. from Section 1.1:	(10, 20, 20, 40, 40, 30, 30, 10)

b. When the boundary temperatures in Fig. (a) are multiplied by 3, the new interior temperatures are also multiplied by 3.

c. The correspondence from the list of eight boundary temperatures to the list of four interior temperatures is a linear transformation. A verification of this statement is not expected. However, it can be shown that the solutions of the steady-state temperature problem here satisfy a superposition principle. The system of equations that approximate the interior temperatures can be written in the form $A\mathbf{x} = \mathbf{b}$, where A is determined by the arrangement of the four interior points on the plate and $\mathbf{b}$ is a vector in $\mathbb{R}^4$ determined by the boundary temperatures.

Problems from § 2.1: 10, 12, 15, 17, 20 ,23

10. $AB = \begin{bmatrix} 2 & -3 \\ -4 & 6 \end{bmatrix}\begin{bmatrix} 8 & 4 \\ 5 & 5 \end{bmatrix} = \begin{bmatrix} 1 & -7 \\ -2 & 14 \end{bmatrix}$, $AC = \begin{bmatrix} 2 & -3 \\ -4 & 6 \end{bmatrix}\begin{bmatrix} 5 & -2 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -7 \\ -2 & 14 \end{bmatrix}$

12. Consider $B = [\mathbf{b}_1 \ \ \mathbf{b}_2]$. To make $AB = 0$, one needs $A\mathbf{b}_1 = \mathbf{0}$ and $A\mathbf{b}_2 = \mathbf{0}$. By inspection of A, a suitable

$\mathbf{b}_1$ is $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$, or any multiple of $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Example: $B = \begin{bmatrix} 2 & 6 \\ 1 & 3 \end{bmatrix}$.

15. False. See the definition of AB.

17. False. The roles of A and B should be reversed in the second half of the statement. See the box after Example 3.

20. True. See Theorem 3(b), read right to left.

23. False. The phrase “in the same order” should be “in the reverse order.” See the box after Theorem 3.

Problems from § 2.2: 1, 5, 7, 9

1. $\begin{bmatrix} 8 & 3 \\ 5 & 2 \end{bmatrix}^{-1} = \frac{1}{16-15}\begin{bmatrix} 2 & -3 \\ -5 & 8 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ -5 & 8 \end{bmatrix}$

5. $\begin{bmatrix} 8 & 3 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} 2 & -3 \\ -5 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

7. The system is equivalent to $A\mathbf{x} = \mathbf{b}$, where $A = \begin{bmatrix} 8 & 3 \\ 5 & 2 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, and the solution is

$\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} 2 & -3 \\ -5 & 8 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 7 \\ -18 \end{bmatrix}$. Thus $x_1 = 7$ and $x_2 = -18$.

Problem 9

a. $\begin{bmatrix} 1 & 2 \\ 5 & 12 \end{bmatrix}^{-1} = \frac{1}{1\cdot 12 - 2\cdot 5}\begin{bmatrix} 12 & -2 \\ -5 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 12 & -2 \\ -5 & 1 \end{bmatrix}$ or $\begin{bmatrix} 6 & -1 \\ -2.5 & .5 \end{bmatrix}$

$\mathbf{x} = A^{-1}\mathbf{b}_1 = \frac{1}{2}\begin{bmatrix} 12 & -2 \\ -5 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 3 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -18 \\ 8 \end{bmatrix} = \begin{bmatrix} -9 \\ 4 \end{bmatrix}$. Similar calculations give

$A^{-1}\mathbf{b}_2 = \begin{bmatrix} 11 \\ -5 \end{bmatrix}, A^{-1}\mathbf{b}_3 = \begin{bmatrix} 6 \\ -2 \end{bmatrix}, A^{-1}\mathbf{b}_4 = \begin{bmatrix} 13 \\ -5 \end{bmatrix}$.

b. $[A \;\; \mathbf{b}_1 \;\; \mathbf{b}_2 \;\; \mathbf{b}_3 \;\; \mathbf{b}_4] = \begin{bmatrix} 1 & 2 & -1 & 1 & 2 & 3 \\ 5 & 12 & 3 & -5 & 6 & 5 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 2 & -1 & 1 & 2 & 3 \\ 0 & 2 & 8 & -10 & -4 & -10 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 & 1 & 2 & 3 \\ 0 & 1 & 4 & -5 & -2 & -5 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 0 & -9 & 11 & 6 & 13 \\ 0 & 1 & 4 & -5 & -2 & -5 \end{bmatrix}$

The solutions are $\begin{bmatrix} -9 \\ 4 \end{bmatrix}, \begin{bmatrix} 11 \\ -5 \end{bmatrix}, \begin{bmatrix} 6 \\ -2 \end{bmatrix}$, and $\begin{bmatrix} 13 \\ -5 \end{bmatrix}$, the same as in part (a).