MATH 200C: Linear Algebra

Some Notes on Assignment 13

§ 2.1: 25, 29, 36; § 2.2: 21, 27, 31; § 2.3: 1, 5, 11, 13, 15, 17, 19, 21

Problems from §2.1

25. Since $\begin{bmatrix} -1 & 2 & -1 \\ 6 & -9 & 3 \end{bmatrix} = AB = [A\mathbf{b}_1 \quad A\mathbf{b}_2 \quad A\mathbf{b}_3]$, the first column of B satisfies the equation $A\mathbf{x} = \begin{bmatrix} -1 \\ 6 \end{bmatrix}$. Row reduction: $[A \quad A\mathbf{b}_1] \sim \begin{bmatrix} 1 & -2 & -1 \\ -2 & 5 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 7 \\ 0 & 1 & 4 \end{bmatrix}$. So $\mathbf{b}_1 = \begin{bmatrix} 7 \\ 4 \end{bmatrix}$. Similarly, $[A \quad A\mathbf{b}_2] \sim \begin{bmatrix} 1 & -2 & 2 \\ -2 & 5 & -9 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -8 \\ 0 & 1 & -5 \end{bmatrix}$ and $\mathbf{b}_2 = \begin{bmatrix} -8 \\ -5 \end{bmatrix}$.

29. Let $\mathbf{b}_p$ be the last column of B. By hypothesis, the last column of AB is zero. Thus, $A\mathbf{b}_p = \mathbf{0}$. However,
$\mathbf{b}_p$ is not the zero vector, because B has no column of zeros. Thus, the equation $A\mathbf{b}_p = \mathbf{0}$ is a linear dependence relation among the columns of A, and so the columns of A are linearly dependent.

36. Since the inner product $\mathbf{u}^T\mathbf{v}$ is a real number, it equals its transpose. That is, $\mathbf{u}^T\mathbf{v} = (\mathbf{u}^T\mathbf{v})^T = \mathbf{v}^T(\mathbf{u}^T)^T = \mathbf{v}^T\mathbf{u}$, by Theorem 3(d) regarding the transpose of a product of matrices and by Theorem 3(a). The outer product $\mathbf{u}\mathbf{v}^T$ is an $n\times n$ matrix. By Theorem 3, $(\mathbf{u}\mathbf{v}^T)^T = (\mathbf{v}^T)^T\mathbf{u}^T = \mathbf{v}\mathbf{u}^T$.

Problems from §2.2

21. (The proof can be modeled after the proof of Theorem 5.) The $n\times p$ matrix B is given (but is arbitrary). Since A is invertible, the matrix $A^{-1}B$ satisfies $AX = B$, because $A(A^{-1}B) = A\,A^{-1}B = IB = B$. To show this solution is unique, let X be any solution of $AX = B$. Then, left-multiplication of each side by A^{-1} shows that X must be $A^{-1}B$: Thus $A^{-1}(AX) = A^{-1}B$, so $IX = A^{-1}B$, and thus $X = A^{-1}B$.

27. Right-multiply each side of $AB = BC$ by B^{-1}, thus $ABB^{-1} = BCB^{-1}$, so $AI = BCB^{-1}$, and $A = BCB^{-1}$.

31. Suppose A is invertible. By Theorem 5, the equation $A\mathbf{x} = \mathbf{0}$ has only one solution, namely, the zero solution. This means that the columns of A are linearly independent, by a remark in Section 1.7.

Problems from §2.3

1. The columns of the matrix $\begin{bmatrix} 5 & 7 \\ -3 & -6 \end{bmatrix}$ are not multiples, so they are linearly independent. By (e) in the IMT, the matrix is invertible. Also, the matrix is invertible by Theorem 4 in Section 2.2 because the determinant is nonzero.

5. $\begin{bmatrix} 0 & 3 & -5 \\ 1 & 0 & 2 \\ -4 & -9 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 3 & -5 \\ -4 & -9 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 3 & -5 \\ 0 & -9 & 15 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 3 & -5 \\ 0 & 0 & 0 \end{bmatrix}$

The matrix is not invertible because it is not row equivalent to the identity matrix.

11. True, by the IMT. If statement (d) of the IMT is true, then so is statement (b).

13. True. If statement (h) of the IMT is true, then so is statement (e).

15. False. Statement (g) of the IMT is true only for invertible matrices.

17. True, by the IMT. If the equation $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution, then statement (d) of the IMT is false. In this case, all the lettered statements in the IMT are false, including statement (c), which means that A must have fewer than n pivot positions.

19. True, by the IMT. If A^T is not invertible, then statement (1) of the IMT is false, and hence statement (a) must also be false.

21. If a square upper triangular $n \times n$ matrix has nonzero diagonal entries, then because it is already in echelon form, the matrix is row equivalent to I_n and hence is invertible, by the IMT. Conversely, if the matrix is invertible, it has n pivots on the diagonal and hence the diagonal entries are nonzero.