MATH 200C: Linear Algebra

Some Notes on Assignment 15

§ 2.3: 23, 31 § 2.4: 5, 9 § 2.5:1, 3, 7

Problems from §2.3: 23, 31

23. If A has two identical columns, this its columns are linearly dependent. Part (e) of the IMTT shows that A cannot be invertible.

31. Statement (b) of the IMT is false for K, so statements (e) and (h) are also false. That is, the columns of K are linearly *de*pendent and the columns do *not* span $\mathbb{R}^n$.

Problems from §2.4: 5, 9

5. Compute the left side of the equation: $\begin{bmatrix} A & B \\ C & 0 \end{bmatrix}\begin{bmatrix} I & 0 \\ X & Y \end{bmatrix} = \begin{bmatrix} AI+BX & A0+BY \\ CI+0X & C0+0Y \end{bmatrix}$

Set this equal to the right side of the equation:

$\begin{bmatrix} A+BX & BY \\ C & 0 \end{bmatrix} = \begin{bmatrix} 0 & I \\ Z & 0 \end{bmatrix}$ so that $\begin{matrix} A+BX=0 & BY=I \\ C=Z & 0=0 \end{matrix}$

Since the (2, 1) blocks are equal, $Z = C$. Since the (1, 2) blocks are equal, $BY = I$. To proceed further, assume that B and Y are square. Then the equation $BY = I$ implies that B is invertible, by the IMT, and $Y = B^{-1}$. (See the boxed remark that follows the IMT.) Finally, from the equality of the (1, 1) blocks, $BX = -A$, $B^{-1}BX = B^{-1}(-A)$, and $X = -B^{-1}A$.

The order of the factors for X is crucial.

9. Compute the left side of the equation:

$$\begin{bmatrix} I & 0 & 0 \\ X & I & 0 \\ Y & 0 & I \end{bmatrix}\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \\ A_{31} & A_{32} \end{bmatrix} = \begin{bmatrix} IA_{11}+0A_{21}+0A_{31} & IA_{12}+0A_{22}+0A_{32} \\ XA_{11}+IA_{21}+0A_{31} & XA_{12}+IA_{22}+0A_{32} \\ YA_{11}+0A_{21}+IA_{31} & YA_{12}+0A_{22}+IA_{32} \end{bmatrix}$$

Set this equal to the right side of the equation: $\begin{bmatrix} A_{11} & A_{12} \\ XA_{11}+A_{21} & XA_{12}+A_{22} \\ YA_{11}+A_{31} & YA_{12}+A_{32} \end{bmatrix} = \begin{bmatrix} B_{11} & B_{12} \\ 0 & B_{22} \\ 0 & B_{32} \end{bmatrix}$

so that $\begin{matrix} A_{11}=B_{11} & A_{12}=B_{12} \\ XA_{11}+A_{21}=0 & XA_{12}+A_{22}=B_{22} \\ YA_{11}+A_{31}=0 & YA_{12}+A_{32}=B_{32} \end{matrix}$.

Since the (2,1) blocks are equal, $XA_{11}+A_{21}=0$ and $XA_{11}=-A_{21}$. Since A_{11} is invertible, right multiplication by A_{11}^{-1} gives $X=-A_{21}A_{11}^{-1}$. Likewise since the (3,1) blocks are equal, $YA_{11}+A_{31}=0$ and $YA_{11}=-A_{31}$. Since A_{11} is invertible, right multiplication by A_{11}^{-1} gives $Y=-A_{31}A_{11}^{-1}$. Finally, from the (2,2) entries, $XA_{12}+A_{22}=B_{22}$. Since $X=-A_{21}A_{11}^{-1}$, $B_{22}=A_{22}-A_{21}A_{11}^{-1}A_{12}$.

Problems from §2.5: 1, 3, 7

1. $L = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -5 & 1 \end{bmatrix}, U = \begin{bmatrix} 3 & -7 & -2 \\ 0 & -2 & -1 \\ 0 & 0 & -1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} -7 \\ 5 \\ 2 \end{bmatrix}$. First, solve $L\mathbf{y} = \mathbf{b}$.

$[L \ \ \mathbf{b}] = \begin{bmatrix} 1 & 0 & 0 & -7 \\ -1 & 1 & 0 & 5 \\ 2 & -5 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -7 \\ 0 & 1 & 0 & -2 \\ 0 & -5 & 1 & 16 \end{bmatrix}$ The only arithmetic is in column 4

$\sim \begin{bmatrix} 1 & 0 & 0 & -7 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 6 \end{bmatrix}$, so $\mathbf{y} = \begin{bmatrix} -7 \\ -2 \\ 6 \end{bmatrix}$.

Next, solve $U\mathbf{x} = \mathbf{y}$, using back-substitution (with matrix notation).

$[U \ \ \mathbf{y}] = \begin{bmatrix} 3 & -7 & -2 & -7 \\ 0 & -2 & -1 & -2 \\ 0 & 0 & -1 & 6 \end{bmatrix} \sim \begin{bmatrix} 3 & -7 & -2 & -7 \\ 0 & -2 & -1 & -2 \\ 0 & 0 & 1 & -6 \end{bmatrix} \sim \begin{bmatrix} 3 & -7 & 0 & -19 \\ 0 & -2 & 0 & -8 \\ 0 & 0 & 1 & -6 \end{bmatrix}$

$\sim \begin{bmatrix} 3 & -7 & 0 & -19 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -6 \end{bmatrix} \sim \begin{bmatrix} 3 & 0 & 0 & 9 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -6 \end{bmatrix}$, so $\mathbf{x} = \begin{bmatrix} 3 \\ 4 \\ -6 \end{bmatrix}$.

To confirm this result, row reduce the matrix $[A \ \mathbf{b}]$:

$$[A\ \ \mathbf{b}] = \begin{bmatrix} 3 & -7 & -2 & -7 \\ -3 & 5 & 1 & 5 \\ 6 & -4 & 0 & 2 \end{bmatrix} \sim \begin{bmatrix} 3 & -7 & -2 & -7 \\ 0 & -2 & -1 & -2 \\ 0 & 10 & 4 & 16 \end{bmatrix} \sim \begin{bmatrix} 3 & -7 & -2 & -7 \\ 0 & -2 & -1 & -2 \\ 0 & 0 & -1 & 6 \end{bmatrix}$$

From this point the row reduction follows that of $[U\ \ \mathbf{y}]$ above, yielding the same result.

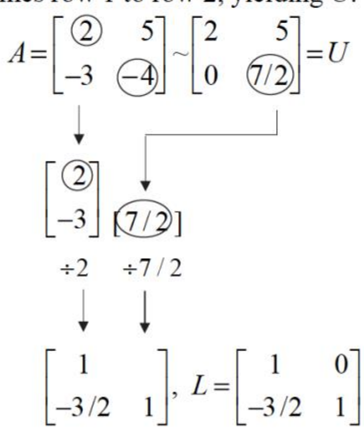
3. $L = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 4 & -1 & 1 \end{bmatrix}, U = \begin{bmatrix} 2 & -1 & 2 \\ 0 & -3 & 4 \\ 0 & 0 & 1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}$. First, solve $L\mathbf{y} = \mathbf{b}$:

$$[L\ \ \mathbf{b}] = \begin{bmatrix} 1 & 0 & 0 & 1 \\ -3 & 1 & 0 & 0 \\ 4 & -1 & 1 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & -1 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 3 \end{bmatrix}, \text{ so } \mathbf{y} = \begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix}.$$

Next solve $U\mathbf{x} = \mathbf{y}$, using back-substitution (with matrix notation):

$$[U\ \ \mathbf{y}] = \begin{bmatrix} 2 & -1 & 2 & 1 \\ 0 & -3 & 4 & 3 \\ 0 & 0 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 2 & -1 & 0 & -5 \\ 0 & -3 & 0 & -9 \\ 0 & 0 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 2 & -1 & 0 & -5 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 2 & 0 & 0 & -2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 3 \end{bmatrix}, \text{ so } \mathbf{x} = \begin{bmatrix} -1 \\ 3 \\ 3 \end{bmatrix}.$$

7. Place the first pivot column of $\begin{bmatrix} 2 & 5 \\ -3 & -4 \end{bmatrix}$ into L, after dividing the column by 2 (the pivot), then add 3/2 times row 1 to row 2, yielding U.



$$A = \begin{bmatrix} 2 & 5 \\ -3 & -4 \end{bmatrix} \sim \begin{bmatrix} 2 & 5 \\ 0 & 7/2 \end{bmatrix} = U$$

$$\begin{bmatrix} 1 & \\ -3/2 & 1 \end{bmatrix}, \quad L = \begin{bmatrix} 1 & 0 \\ -3/2 & 1 \end{bmatrix}$$