

MATH 224: Vector Calculus

Some Notes on Class 1

February 9, 2026

CALCULUS: Limits, Derivatives, Integrals of Functions

Classic Setting: $y = f(x)$ where $f : \mathcal{R}^1 \rightarrow \mathcal{R}^1$

Input and Output Are Each Single Numbers

Graph is a CURVE (1 Dimensional) in Plane (\mathcal{R}^2)

Idea of a function Generalizes Easily:

Input: One Object

Output: One Object

**VECTOR is unifying concept of Linear Algebra and
Multivariable Calculus**

Calculus I and II: Real-Valued Function of Real Variable

Multivariable Calculus: Vector-Valued Function of Vector Variable

Ultimate Goal: $f : \mathcal{R}^n \rightarrow \mathcal{R}^m$.

Example: Let A be a 3×4 matrix and \mathbf{x} a 4×1 vector.

Consider the function $f(\mathbf{x}) = A\mathbf{x}$

$$\begin{pmatrix} - & - & - & - \\ - & - & - & - \\ - & - & - & - \end{pmatrix} \begin{pmatrix} - \\ - \\ - \\ - \end{pmatrix} = \begin{pmatrix} - \\ - \\ - \end{pmatrix}$$

3×4 4×1 3×1

A \mathbf{x} $A\mathbf{x}$

Classic Applications of Calculus

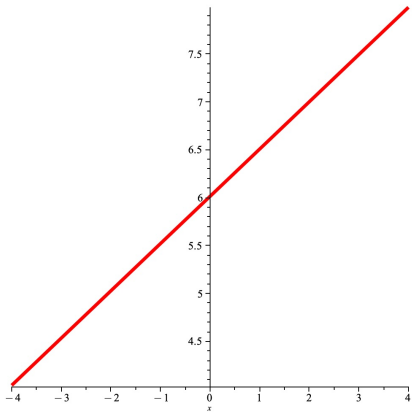
- ▶ Motion of Object Along Straight Line
(Position, Velocity, Acceleration)
- ▶ Profit as a Function of Price
- ▶ Amount of Drug in Bloodstream at time t

Real World Is Much More Complicated

- ▶ Motion: Objects Move in Plane or Space
Need Vector to Describe Location
- ▶ Profit: Depends on Prices, Demand, Taxes, Production Costs
- ▶ GPA: Function of Many Course Grades

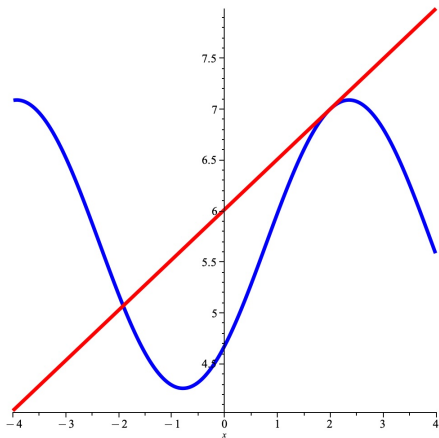
INPUT: Middlebury College Budget

OUTPUT: Amount for Financial Aid, Staff Salaries, Dining
Services, etc.



Red Line L has slope m and passes through $(2,7)$

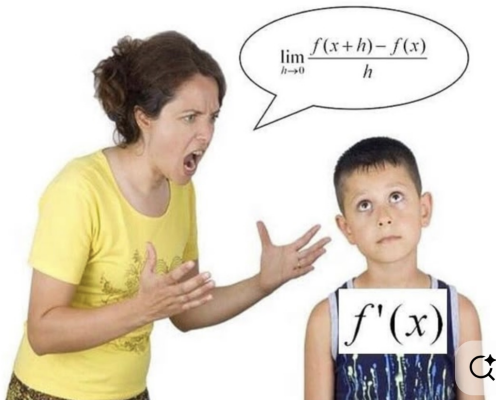
What is an equation for the line?



Red Line L has slope m and passes through $(2,7)$

$$y = f(x), \quad f(2) = 7$$

When your mom calls you by your full name:



Definition of Derivative

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \text{ IF THE LIMIT EXISTS}$$

Example: Find $f'(x)$ if $f(x) = x^3$

Solution:

$$\begin{aligned} f(x+h) - f(x) &= (x+h)^3 - x^3 \\ &= x^3 + 3x^2h + 3xh^2 + h^3 - x^3 \\ &= 3x^2h + 3xh^2 + h^3 \\ &= h [3x^2 + 3xh + h^2] \end{aligned}$$

$$\text{so } \frac{f(x+h) - f(x)}{h} = [3x^2 + 3xh + h^2] \text{ if } h \neq 0$$

$$\text{Hence } f'(x) = \lim_{h \rightarrow 0} 3x^2 + 3xh + h^2 = 3x^2$$

Example: Determine $g'(x)$ if $g(x) = x f(x)$ where f is a differentiable function.

Solution:

$$\begin{aligned}g(x+h) - g(x) &= (x+h)f(x+h) - xf(x) \\ &= xf(x+h) + hf(x+h) - xf(x) \\ &= x[f(x+h) - f(x)] + hf(x+h)\end{aligned}$$

Thus The Difference Quotient is

$$\frac{g(x+h) - g(x)}{h} = x \frac{f(x+h) - f(x)}{h} + f(x+h)$$

Taking Limits as $h \rightarrow 0$:

$$\frac{f(x+h) - f(x)}{h} \rightarrow f'(x)$$

$f(x+h) \rightarrow f(x)$ since f is continuous

$$\text{Hence } g'(x) = xf'(x) + f(x)$$